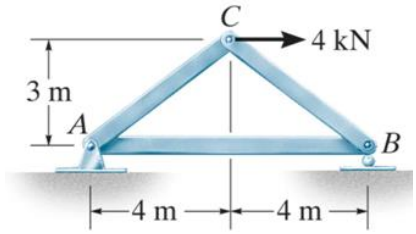
EXAMPLES
4.1: Virtual Work (Trusses)

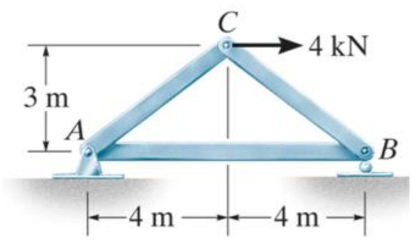
Example #1

Given that for all members: A = 400 mm2 and E = 200 GPa, find the vertical displacement of joint C

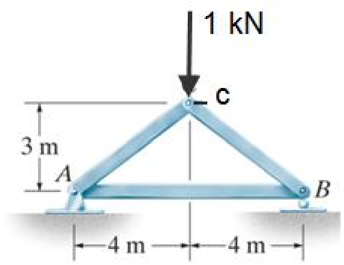


1. Truss determinate?

2. Structure subjected to real forces



3. Structure subjected to virtual forces



4. Apply virtual work equation: $1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$

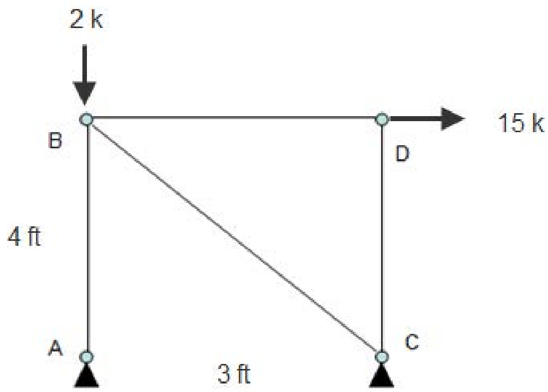
Member	A	L	N	n	nNL
AB					
AC					
CB					
				Sum	**10.67**

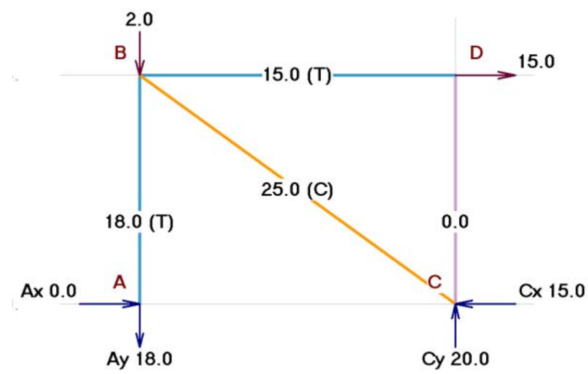
$$1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$$

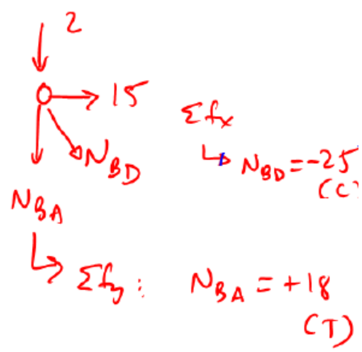
Example #2

Determine the horizontal displacement of joint B using **Method of Virtual Work**.

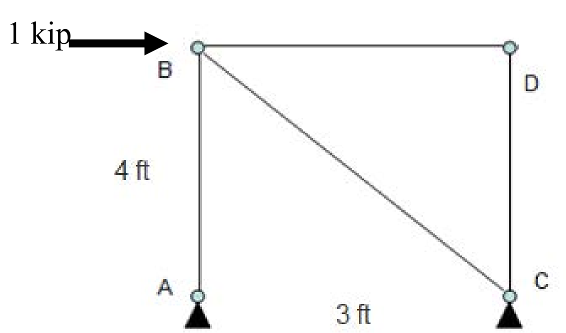
Truss analysis #1: real system = actual loads

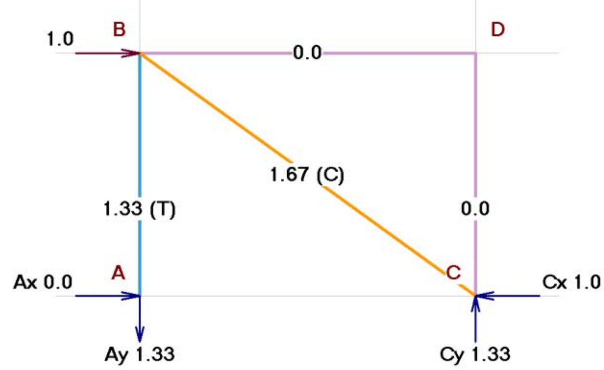


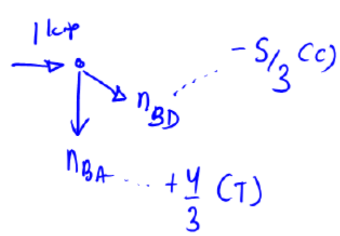




Truss analysis #2: virtual system = unit load in direction of displacement

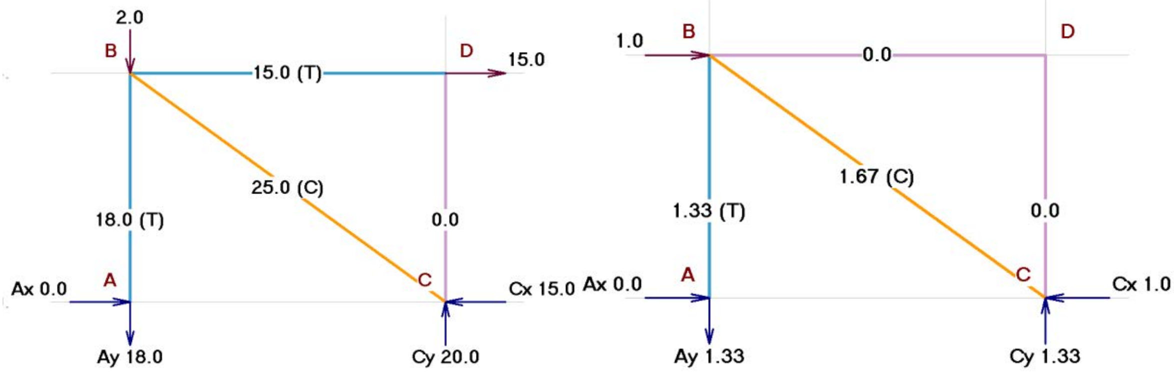






Summary:

Real forces (N) Virtual forces (n)



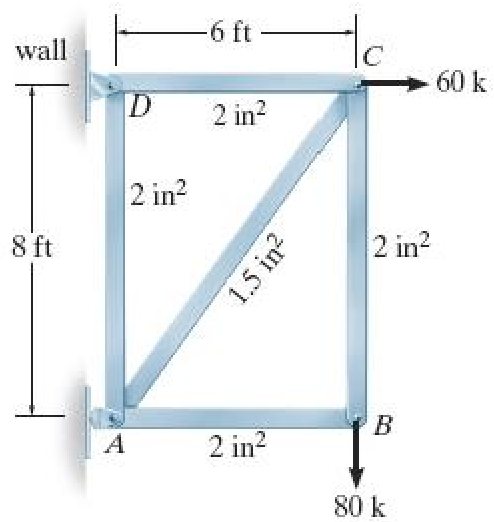
Member	L (in)	A (in^2)	N (kip)	n (kip)	nNL (kip^2*in)
AB	48	3			1152
BC	60	3			2500
BD	36	3			0
CD	48	3			0
				Sum	**3652**

Let's work in [Kip, in.] :

$$1\ \ kip * \Delta = \sum n \cdot \frac{NL}{AE} = \frac{3652 kip^2 * in}{3in^2 \times \left(29 \times 10^3 kip / in^2\right)} \rightarrow \Delta = 0.042\ in.$$

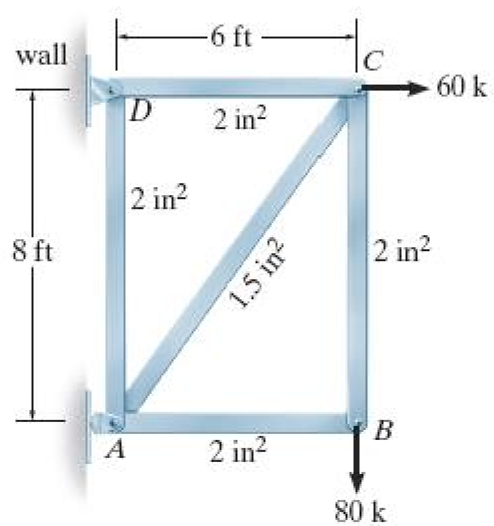
Example #3

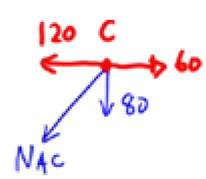
Determine the vertical displacement of joint C of the steel truss shown in Fig. 9–10a. Due to radiant heating from the wall, member AD is subjected to an *increase* in temperature of $\Delta T = +120°F$. Take $\alpha = 0.6(10^{-5})/°F$ and $E = 29(10^3)$ ksi. The cross-sectional area of each member is indicated in the figure.

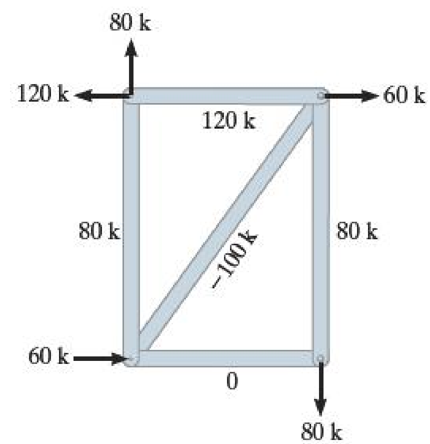


1. Truss determinate?

2. Structure subjected to real forces

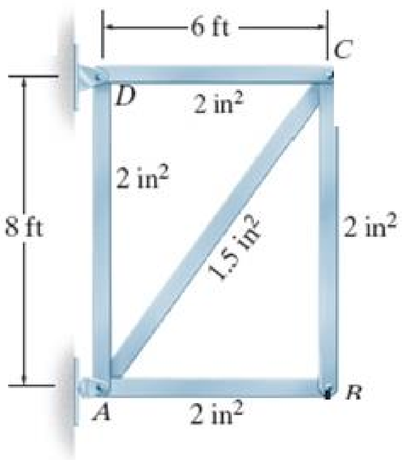


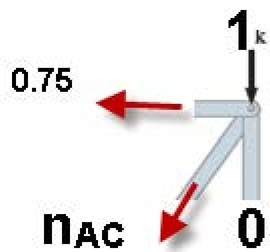


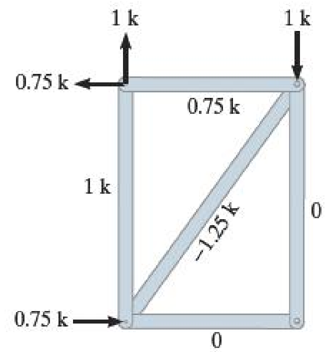


real forces **N**

3. Structure subjected to virtual forces







virtual forces **n**

4. Apply virtual work equation: $1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$

Member	A (in^2)	L (in)	N (kip)	n (kip)	nNL (kip * in^2)	nNL/A	ΔT	$n \cdot \alpha \Delta T L$
AB	2	72	0	0				
AC	1.5	120	-100	-1.25				
AD	2	96	80	1				
BC	2	96	80	0				
CD	2	72	120	0.75				

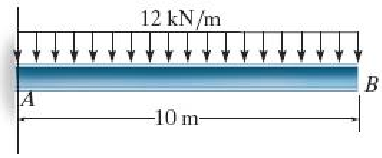
Let's work in [Kip, in.] :

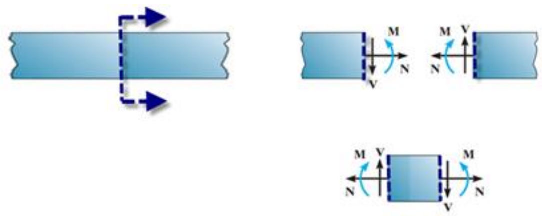
$$1 \cdot \Delta = \sum n \cdot \frac{NL}{AE} + \sum n \cdot \alpha \Delta T L + \sum n \cdot \Delta L$$

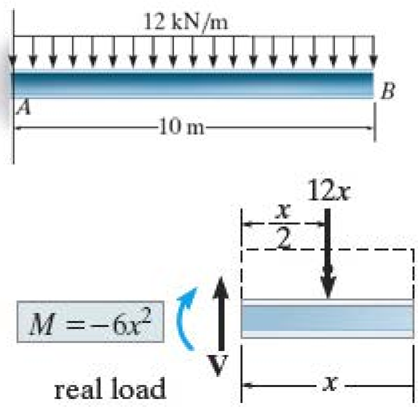
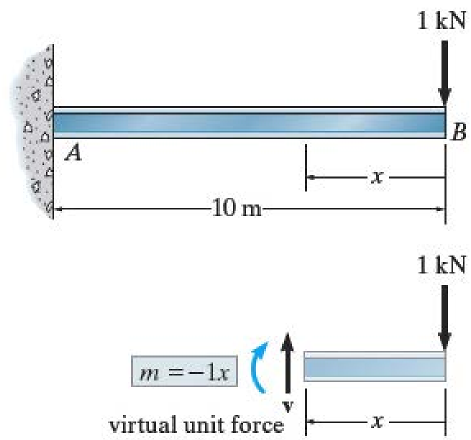
EXAMPLES
4.2: Virtual Work Beams
(using Direct Integration)

Example 1: Use Virtual work (by direct integration) to find vertical displacement @ B

$E = 200\ \text{GPa},\ I = 500(10^6)\ \text{mm}^4.$





Structure subjected to real forces	Structure subjected to virtual unit load
	

Apply virtual work equation: $1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$

$$1\ \text{kN} \cdot \Delta_B = \int_0^L \frac{mM}{EI} dx = \int_0^{10} \frac{(-1x)(-6x^2)\ dx}{EI}$$

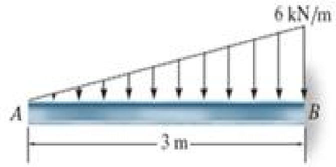
$$1\ \text{kN} \cdot \Delta_B = \frac{15(10^3)\ \text{kN}^2 \cdot \text{m}^3}{EI}$$

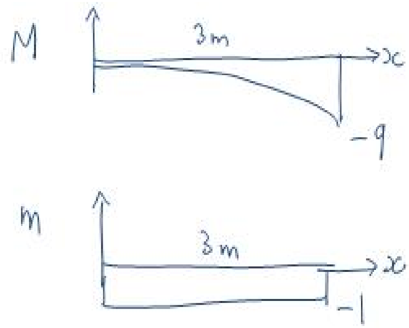
$$\Delta_B = \frac{15(10^3)\ \text{kN} \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2(500(10^6)\ \text{mm}^4)(10^{-12}\ \text{m}^4/\text{mm}^4)}$$

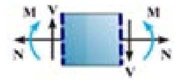
$$= 0.150\ \text{m} = 150\ \text{mm}$$ ***Ans.***

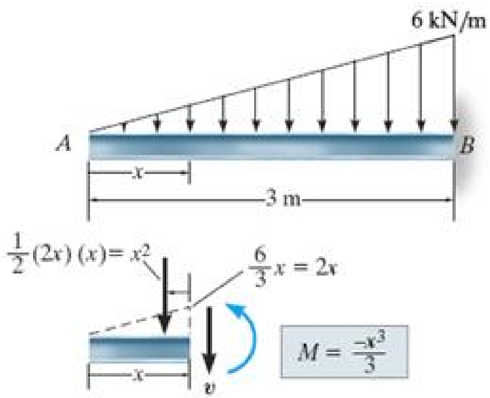
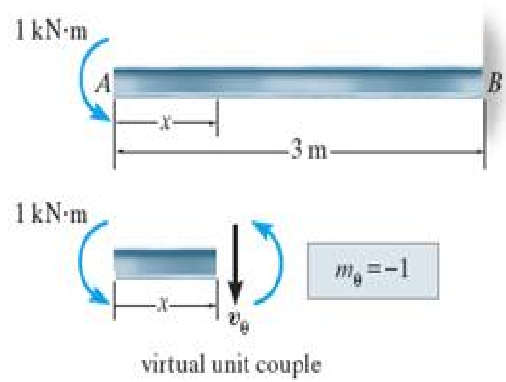
Example 2: Use Virtual work (by direct integration) to find rotation (slope) at point A, Θ_A

$E = 200$ GPa, $I = 60(10^6)$ mm^4.







Structure subjected to real forces	Structure subjected to virtual unit load
	

Apply virtual work equation:

$$(1\,\text{kN}\cdot\text{m})\cdot\theta_A = \int_0^L \frac{m_\theta M}{EI}dx$$

$$= \int_0^3 \frac{(-1)\left(\frac{-x^3}{3}\right)}{EI}dx$$

$$= \frac{1}{3EI}\int_0^3 x^3\,dx$$

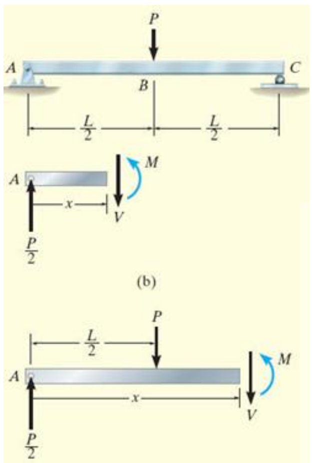
$$= \frac{6.75}{200(10^6)\,\text{kN/m}^2(60(10^6)\,\text{mm}^4)(10^{-12}\,\text{m}^4/\text{mm}^4)}$$

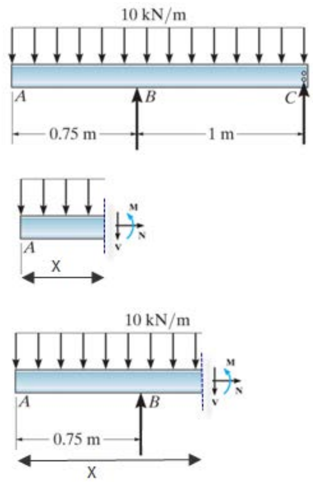
$\theta_A = 0.000563$ rad **Ans.**

Note: sometimes when using direct integration will need <u>several cuts</u> to define <u>M(x)</u>

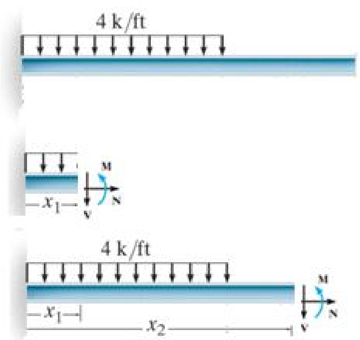
M(x) will have <u>discontinuities</u> when we have variation in loading along the beam:

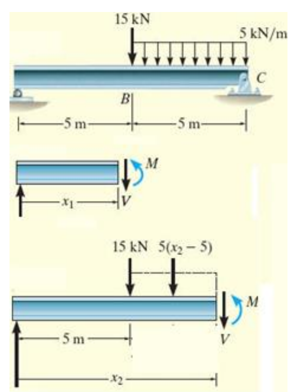
- If there is a **concentrated load** (reaction or external load)**,** we will have to make a cut <u>before</u> and <u>after</u> the concentrated load



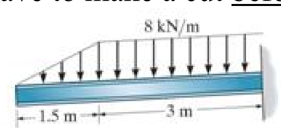


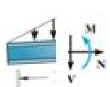
- If there is a **distributed load** on only part of the beam**,** we will have to make a cut <u>inside</u> and <u>before/after</u> the distributed load

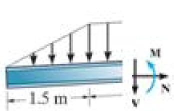




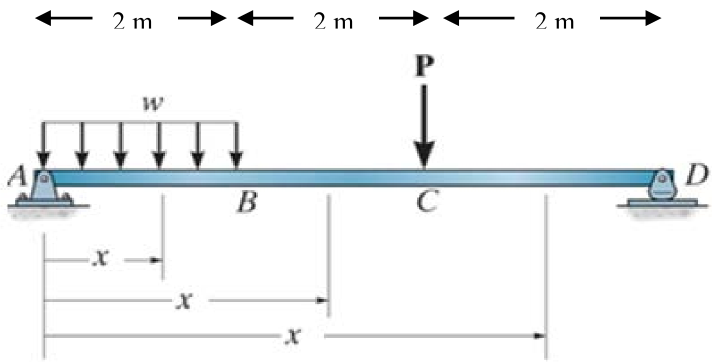
- If there is a **distributed load** and the slope of the distributed loading changes**,** we will have to make a cut <u>before</u> and <u>after</u> the point of change







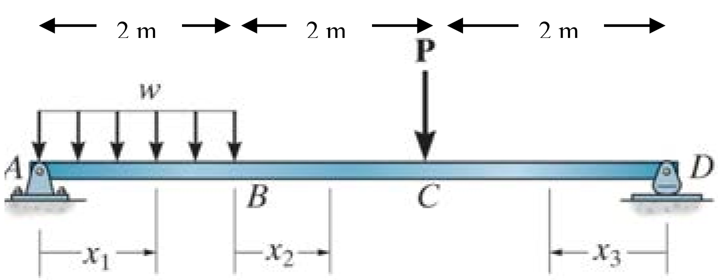
We can use the same origin point for all the cuts, we just need to keep in mind what we choose when we define the interval for which M(x) is valid:



(Hibbeler, 2007)

M1(x) ….valid between $0 \leq x \leq 2$
M2(x) ….valid between $2 \leq x \leq 4$
M3(x) ….valid between $4 \leq x \leq 6$

Or we may use different origin points in order to simplify calculations.

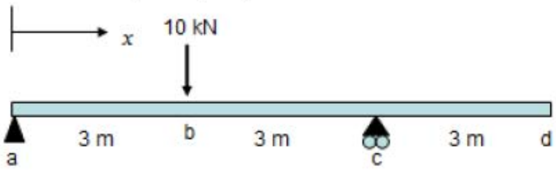


(Hibbeler, 2007)

M1(x1) ….valid between $0 \leq x1 \leq 2$
M2(x2) ….valid between $0 \leq x2 \leq 2$
M3(x3) ….valid between $0 \leq x3 \leq 2$

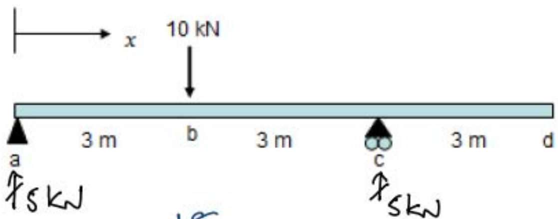
EXAMPLE #3 (with details)

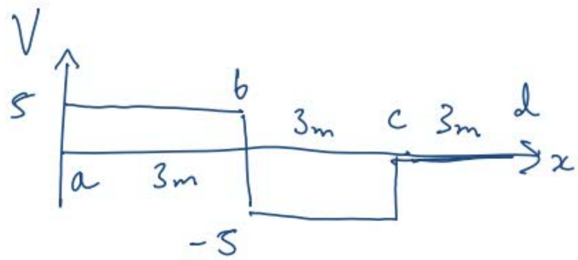
Use virtual work to calculate the **deflection** and **slope** at point "d" of the beam
E = 200 GPa , I= 50(10^6) mm^4.

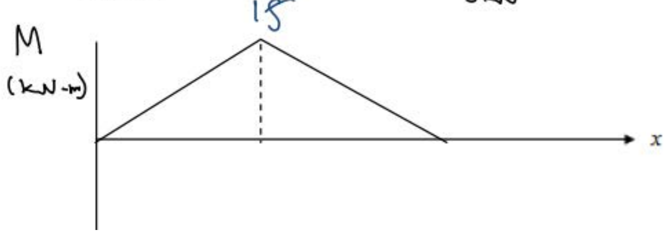


1. Beam determinate?

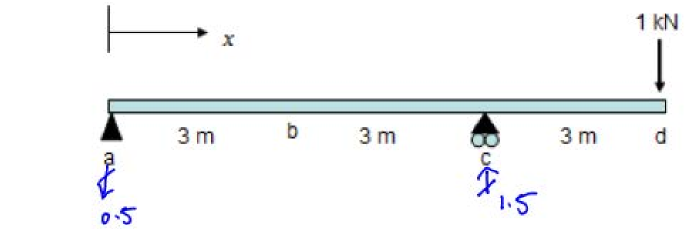
2. Structure subjected to real forces

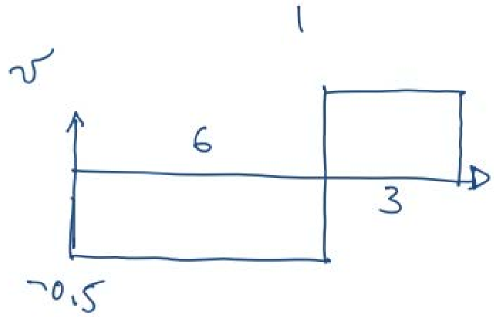


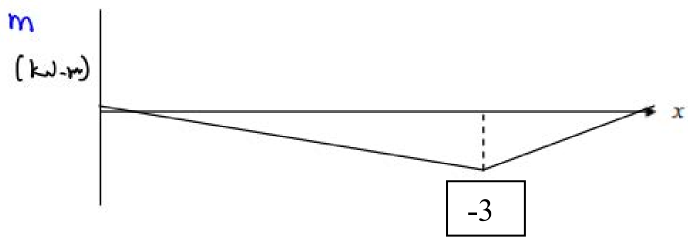




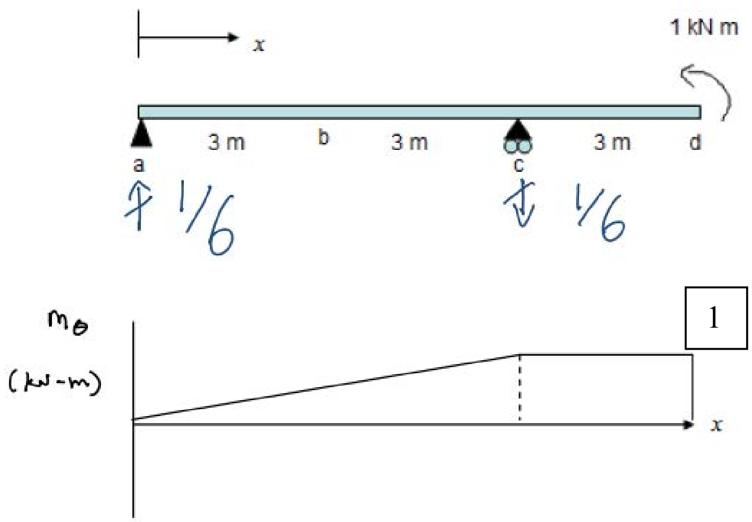
3. Structure subjected to virtual unit load at point “d”







4. Structure subjected to virtual unit couple



$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

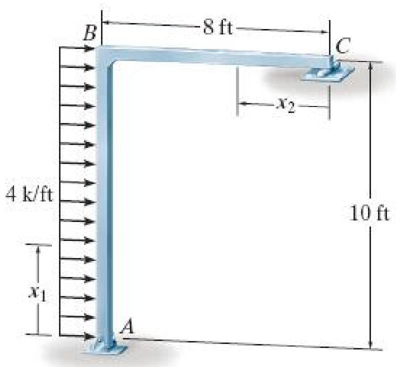
==

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

Example: Frame

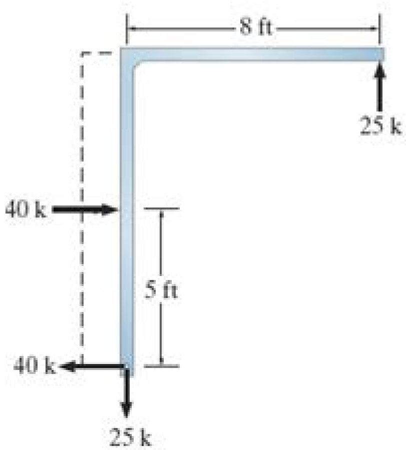
Use virtual work (**by direct integration**) to calculate the horizontal displacement at joint C of the frame given that that E= 29(10^3) and I = 600 in^4 *(EI* is constant for all members)

Frame determinate?

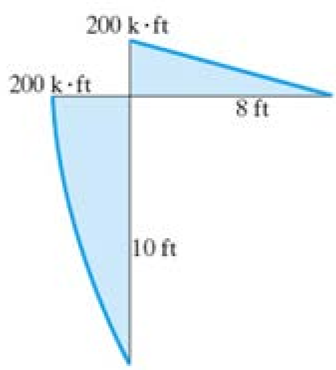


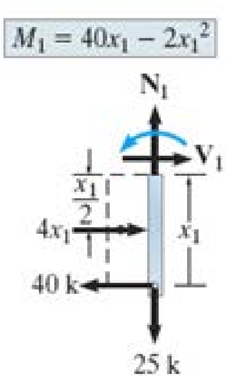
Structure subjected to **real forces**:

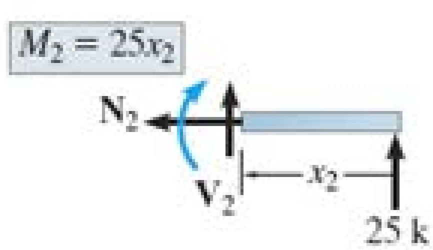
Loads



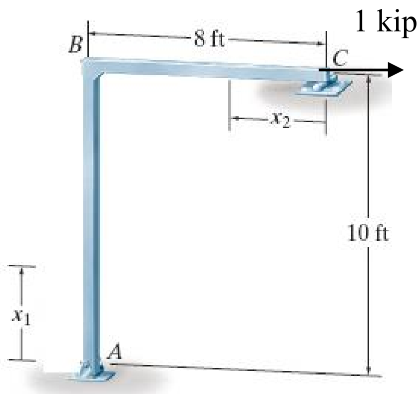
Moment



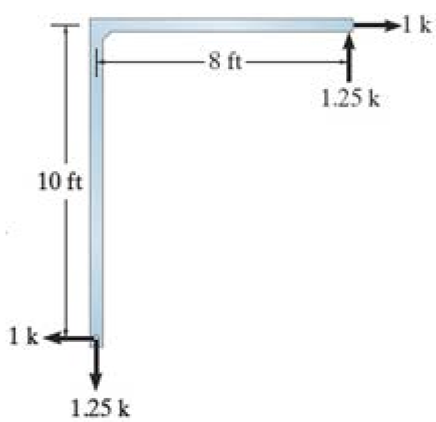




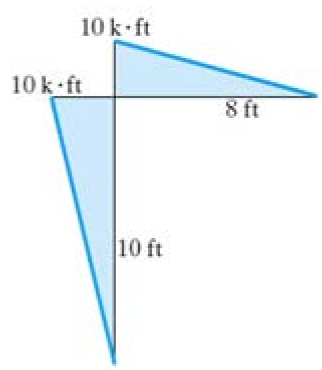
Structure subjected to **virtual unit load**:

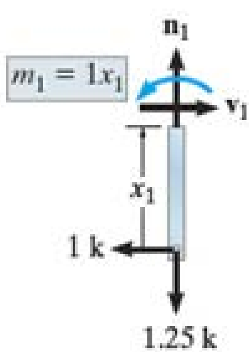


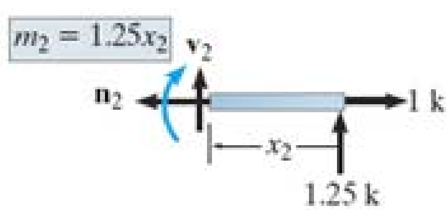
Loads



Moment







Apply virtual work equations:

Virtual-Work Equation.

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} dx =$$

$$\Delta_{C_h} = \int_0^{10} \frac{(\qquad)(\qquad)}{EI} dx_1 + \int_0^{8} \frac{(\qquad)(\qquad)}{EI} dx_2$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI}$$

$$= \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{EI}$$

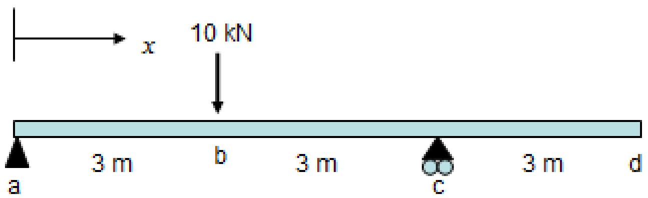
$$\Delta_{C_h} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{[29(10^3) \text{ k/in}^2((12)^2 \text{ in}^2/\text{ft}^2)][600 \text{ in}^4(\text{ft}^4/(12)^4 \text{ in}^4)]}$$

$$= 0.113 \text{ ft} = 1.36 \text{ in.}$$

EXAMPLES
4.2: Virtual Work Beams PART II
(Graphical method / Mohr's method)

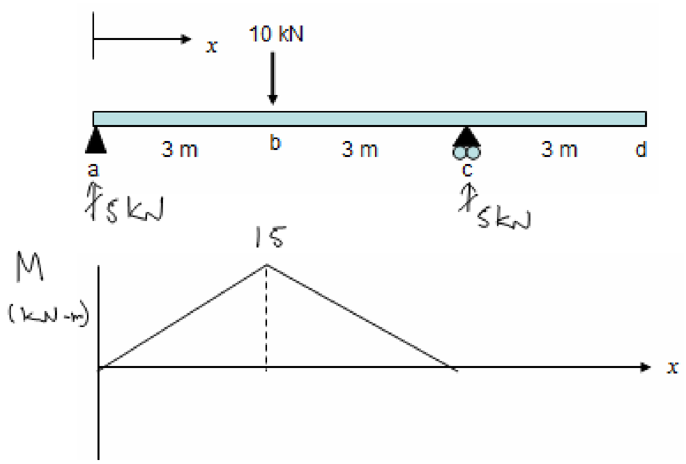
Example : Use virtual work to calculate the deflection at point "d" of the beam shown given that E =200 GPa and I = 50(10^6) mm^4.

Note: Use "geometric methods" or "mohr's tables" to compute $\int_0^L mMdx$

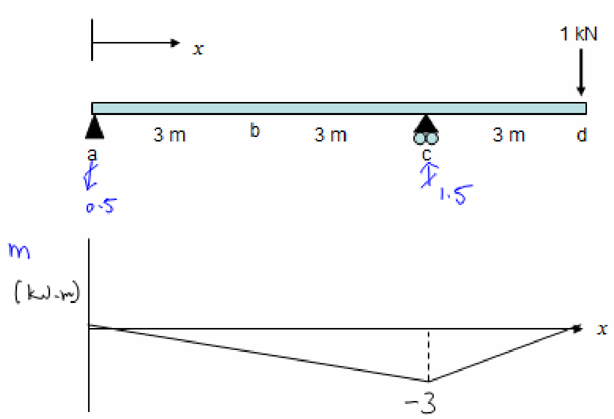


Recall we calculated the moment diagrams previously:

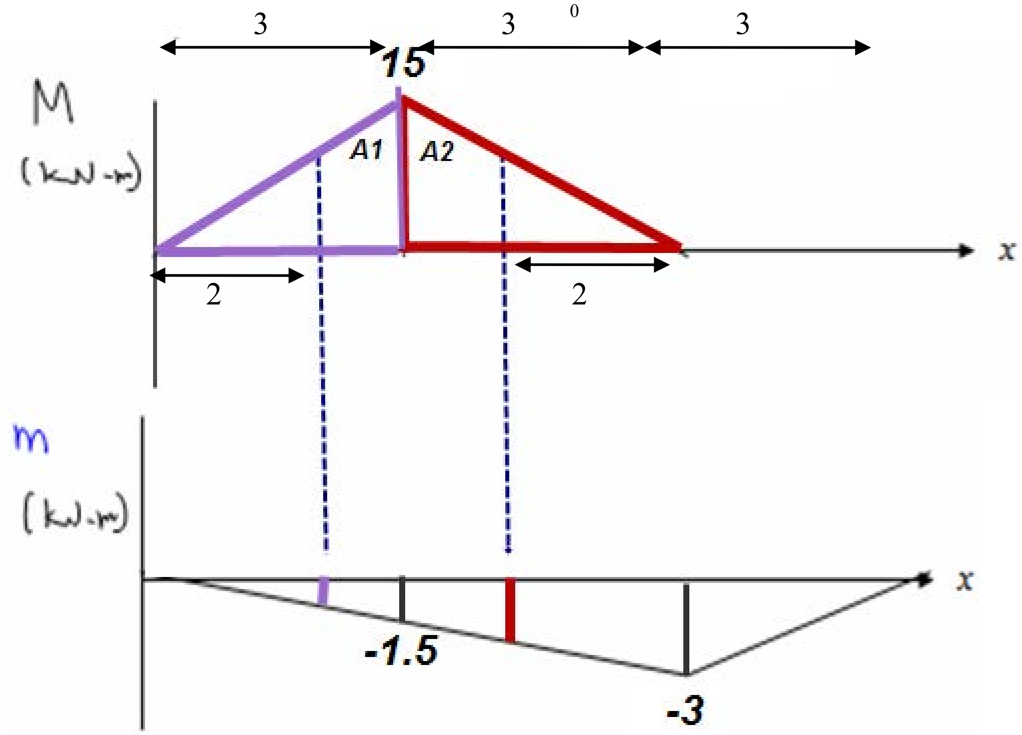
Structure subjected to real forces



Structure subjected to virtual unit load at point "d"



OPTION 2A: Using geometric method to compute $\int_0^L mM dx$

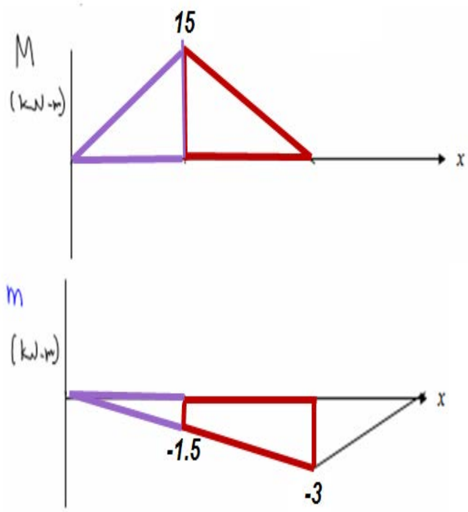


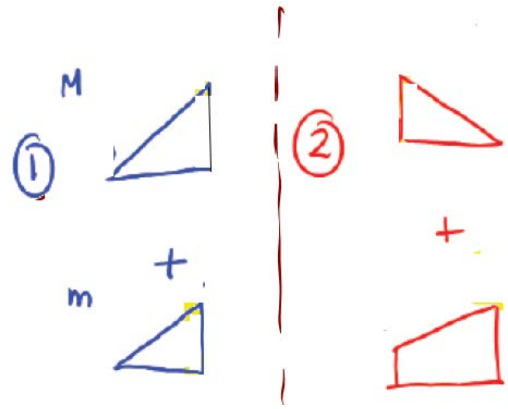
Area			
1			-22.5
2			-45
3			0
		$\sum$	-67.5

Apply virtual work equations:

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

OPTION 2B: Using Mohr's tables to compute $\int_0^L mM\,dx$





Area	
1	-22.5
2	-45
3	0
$\sum$	-67.5

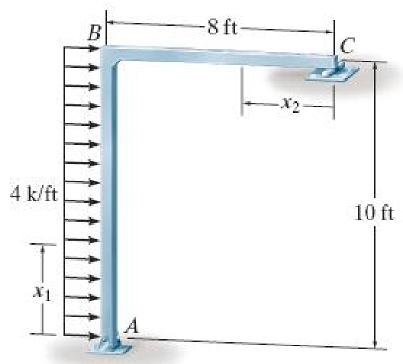
Apply virtual work equations:

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$

Example: Frame

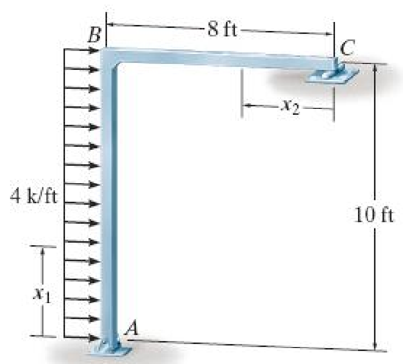
Use virtual work to calculate the horizontal displacement at joint C of the frame given that that *E*= 29(10^3) and *I* = 600 in^4 *(EI* is constant for all members)

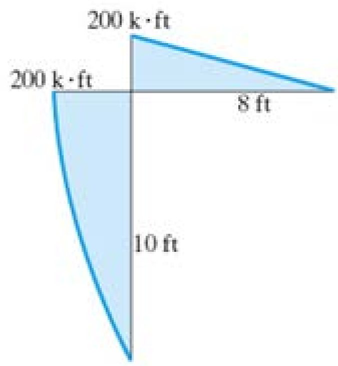
Note: Use “geometric methods” or “mohr's tables” to compute $\int_0^L mM dx$



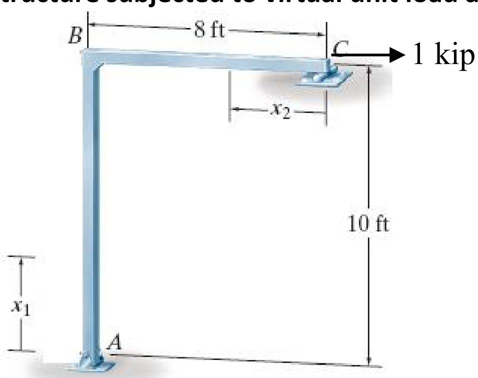
Recall we calculated the moment diagrams previously:

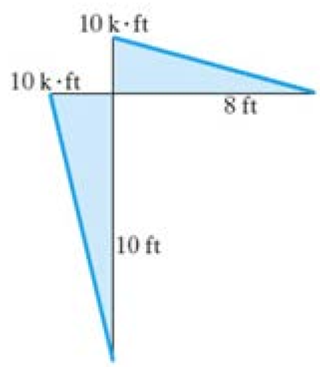
Structure subjected to real forces



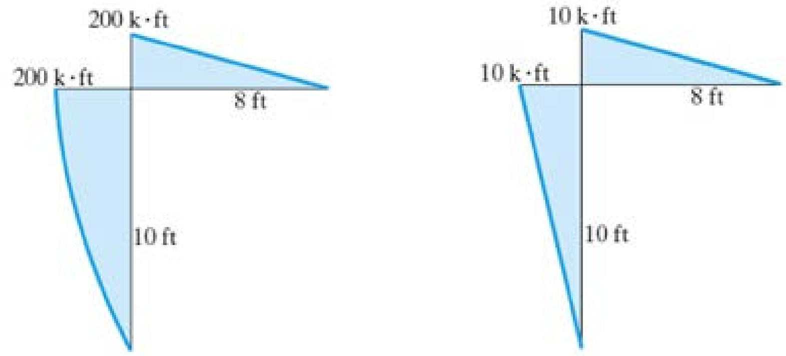


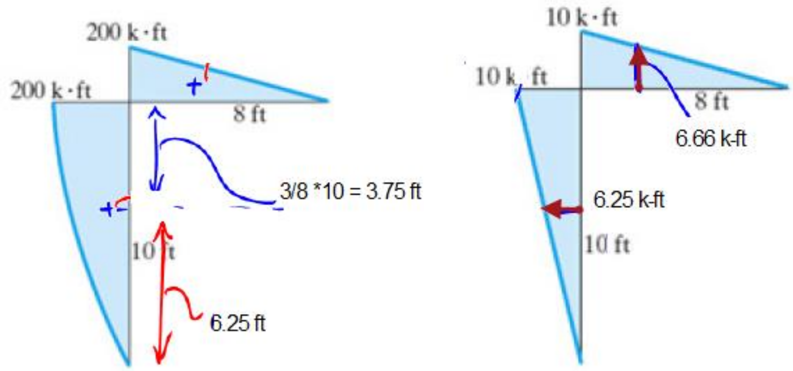
Structure subjected to virtual unit load at point “c”





<u>OPTION 2A:</u> Using <u>geometric method</u> to compute $\int_0^L mM dx$

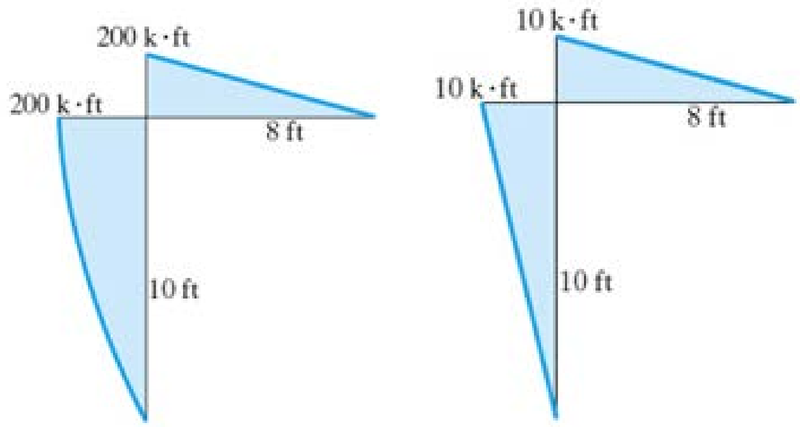




Area			
1			8333
2			5330
		$\sum$	13666

Apply virtual work equations: $1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$

<u>OPTION 2B:</u> Using <u>Mohr's tables</u> to compute $\int_0^L mM dx$



Area	
1	
2	
$\sum$	13666

Apply virtual work equations: $1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$

$$1\text{ kip} \times \Delta = \frac{13666}{EI} \quad [k, ft]$$

$$= \frac{13\,666}{\left(29 \times 10^3 \frac{kip}{in^2} \cdot 12^2\right)\left(600 in^4 * \frac{1}{(12)^4}\right)}$$

=0.113 ft = 1.36 in.