Lecture 8.1: Slope deflection method

Lecture outline:

1. Kinematic Indeterminacy
2. Displacement-based methods of analysis
3. Slope-deflection method
4. Slope-deflection equations
5. Fixed-end moments
6. Equilibrium equations
7. Procedure

1. Kinematic Indeterminacy

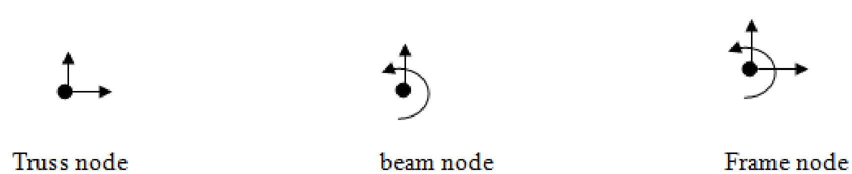
Static indeterminacy = # of excess "redundant" forces to solve for all unknown forces/RXs.

- if SI= 0 we can solve the structure strictly using equilibrium
- if SI = n > 0, we need to define "n" compatibility equations

Kinematic indeterminacy # of extra displacements to define complete displaced shape of structure.

This is related to the concept of **Degrees of Freedom (DOF)**

- D.O.F. = the total number of displacements permitted at a joint (node)
  - A truss has **two** DOFs per joint (two *linear* displacements)
  - A beam has **two** DOFs per joint (one *linear* displacement and one *angular* displacement)
  - A frame has **three** DOFs per joint (two *linear* displacement and one *angular* displacement)



- We can subdivide DOFs into **two categories**:
  - Constrained Degrees of Freedom (CDOF)
    - DOFs = 0 (i.e. displacements we know are equal to zero)
    - (related to support conditions)
- Unconstrained Degrees of Freedom (UDOF)
    - DOFs ≠ 0 (i.e. unknown joint displacements)
- Degree of Kinematic indeterminacy → **KI = total # of UDOF in the structure**

2. Displacement-based methods of analysis

There are two main classical methods that can be used to solve indeterminate structures: force-based methods and displacement-based methods.

In the "force-based" methods, we use the following steps:

(1) we start by identifying the degree of *static indeterminacy* of the structure (i.e. redundant forces)
(2) we then write force-displacement relationships
(3) we then write <u>compatibility equations</u>
(4) finally we solve for the **unknown redundant forces** using these equations

In the "displacement-based" methods, we use the following steps:

(1) we start by identifying the degree of *kinematic indeterminacy* of the structure
(2) we then we write force-displacement relationships
(3) we then write <u>equilibrium equations</u>
(4) we solve for the **unknown displacements** using these equations

We can list the following displacement based methods:

- slope-deflection method (Ch. 11 - Hibbeler)
- moment distribution method (Ch. 12 - Hibbeler)
- matrix format: Stiffness method (Ch. 15 - Hibbeler)

	Force method	***Slope-deflection method***
unknowns	Forces	Displacements
Equations used	Compatibility equations	Force-displacement relationships (slope-deflection equations)
Methodology	Based on static indeterminacy: • Redundancy • Based on "choosing" redundant (s) • Element of **choice**	Based on kinematic indeterminacy: • Degrees of freedom (D.O.F.) • no element of choice
If put in matrix format	Flexibility coefficients • Flexibility method	Stiffness coefficients • Stiffness method

3. Slope-deflection method

The Slope-deflection method is a classical analysis method which can be used to analyse both indeterminate beams and frames (a variation of the method can be used in the analysis of trusses).
An understanding of this classical method is important as it is the basis of the **stiffness method**.

This method considers **displacements** as the primary unknowns

Basic steps:

1. One begins by identifying the constrained and unconstrained **degrees of freedom** in the structure.

2. Then one writes **force-displacement expressions** relating the *member end-moments* to the *unknown displacements*
    - These are called "slope-deflection" equations

3. Then one writes **equilibrium** equations at the member joints

4. Then one combines the **force-displacement relationships** and **equilibrium equations** to solve for the *unknown displacements*.

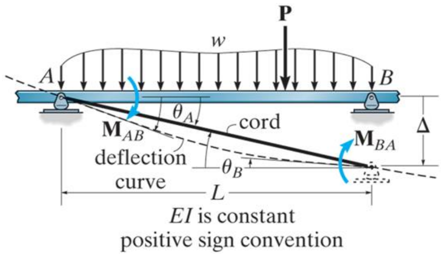
5. Once the displacements are known we use the **force-displacement relationships** in *step 2* to solve for the *member end-moments.*

6. Finally, since the structure is now reduced to a determinate structure, one can *find the reactions, shear* and *moment diagrams*.

4. Slope-deflection equations

For **each member** we apply a general expression called the **slope-deflection equation** to relate the **moments at each joint** to the **unknown displacements** that occur there.

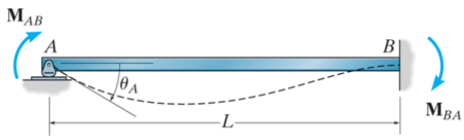
Consider member AB in the continuous beam below:



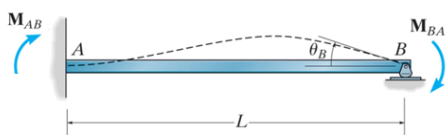
We find the moments at joint A (***M_{AB}***) and joint B (***M_{BA}***), due to:

- angular displacements (i.e. slopes θ)
- relative linear displacements (Δ)
- external loads

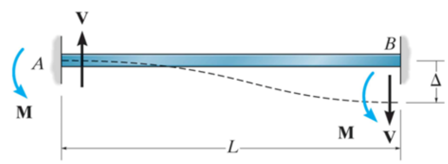
Effect 1: angular displacement at A:



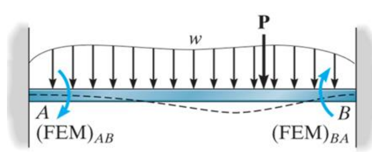
Effect 2: angular displacement at B:



Effect 3: relative linear displacement between A and B:



Effect 4: external loads:



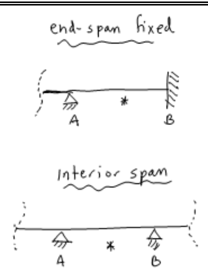
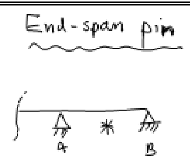
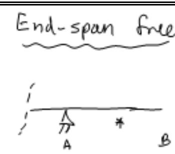
If we add up all the effects using superposition we obtain the **General** **slope-deflection equations:**

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + FEM_A$$

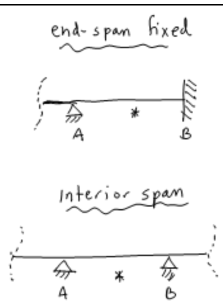
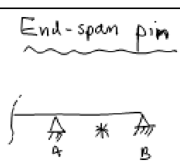
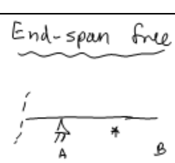
$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + FEM_B$$

Eq #1: $M_N = 2E\left(\frac{I}{L}\right)\left[2\theta_N + \theta_F - 3(\Psi)\right] + FEM_N$

Eq #2: $M_N = 3E\left(\frac{I}{L}\right)\left[\theta_N - (\Psi)\right] + FEM_N$

3 possible cases		
	 (end –span w/ far-end pin/ or roller)	 (end –span w/ far-end free)

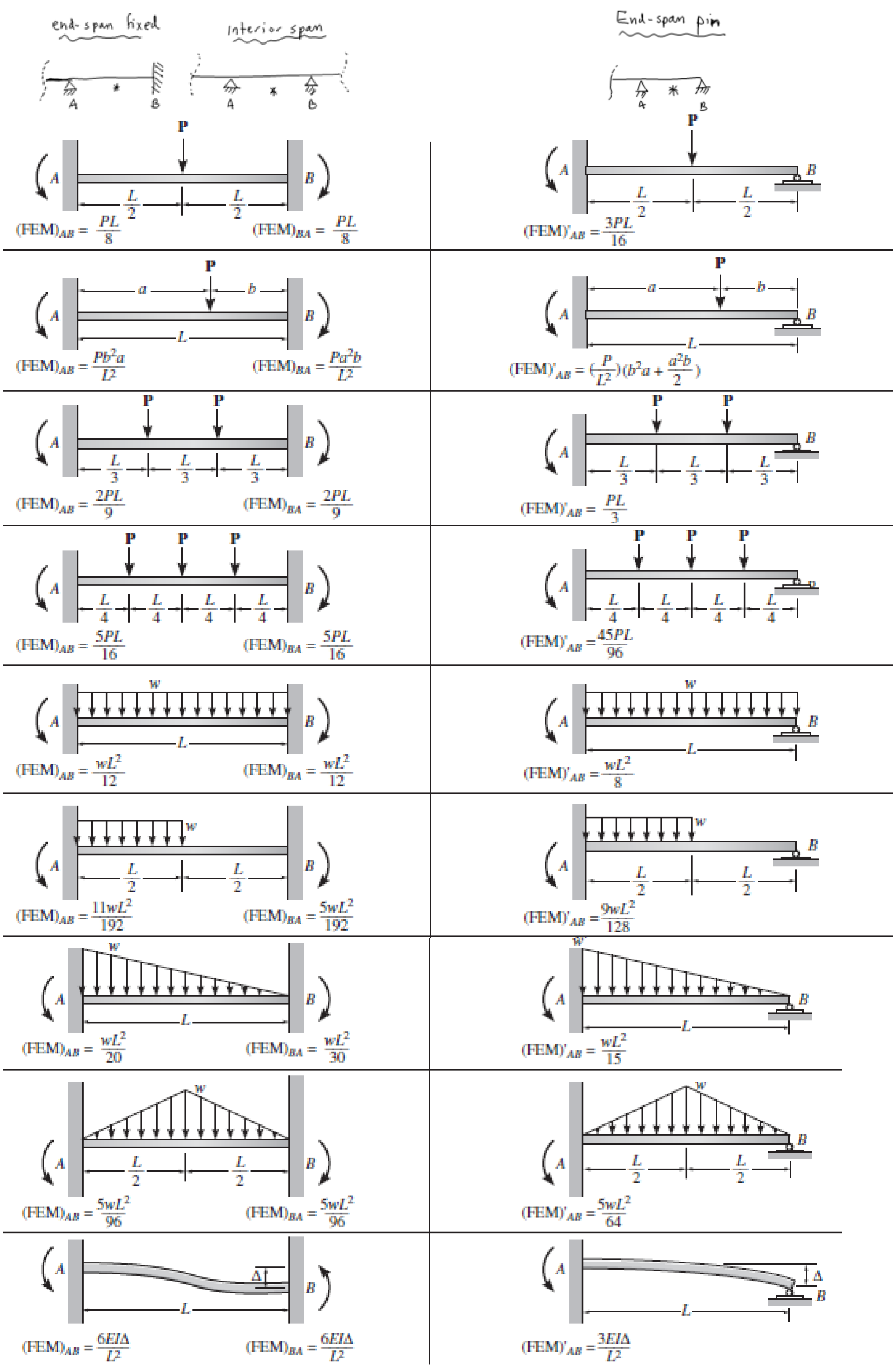
* ****note we draw all moments as clockwise = positive (+) *****

	Case1: **Interior span** or **end-span fixed** • For **internal spans** → **Eq.1** for **both ends** • For **end spans** with **far end fixed** → **Eq.1** for **both ends** • Example: M_{AB} .... A = near, B = far $M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + FEM_A$ M_{BA} .... B = near, A = far $M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + FEM_B$
	Case2: **end-span pin/roller** • For **end spans** with **far end pin/roller** → **Eq.2** for **near end** only ○ Take the interior support as the "N" (near) M_{AB} .... A = near, B = far $M_{AB} = 3E\left(\frac{I}{L}\right)\left[\theta_A - \left(\frac{\Delta}{L}\right)\right] + FEM_A$
	Case3: **end-span free** For **end spans** with **far end free** do not write slope-defl. equations (instead we use statics)

5. Fixed-end moments

* ****FEM** is **positive (+)** if **clockwise** ****

Interior span or End-span fixed | **End-span w/ pin or roller**



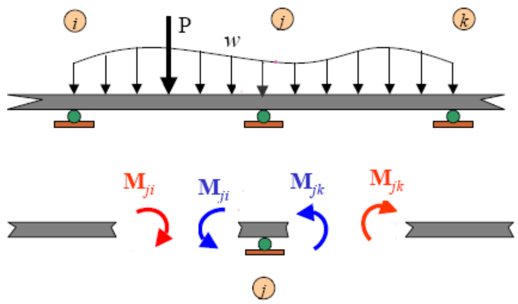
6. Equilibrium equations

The next step is to write a set of **equilibrium equations**.

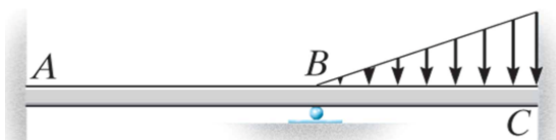
- ... for beams write equilibrium equations at the **interior supports**

How? .... Draw a **FBD** of the joint using a very small segment:

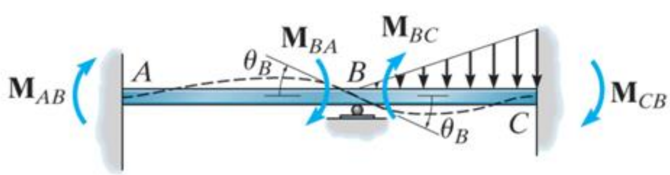
- the sum of the **moments at the joint** has to be zero (if there is no external couple).
- we draw the moments **counter-clockwise** because of the action-reaction principle



Example:

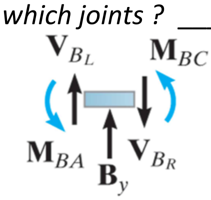


Slope-deflection equations



- Gives me _______ equations with ___________ unknowns
- Notice: moments and rotations are drawn **clockwise**

Equilibrium at which joints ? _______



- Notice: moments in small joint segment are drawn **counter-clockwise**
- Gives me _______ equation (s)

- Total # of equations = _____________ = # of unknowns

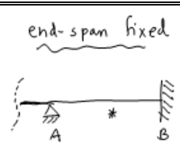
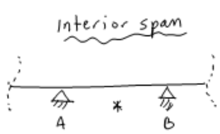
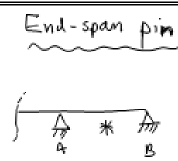
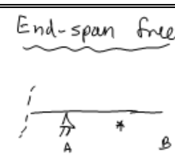
Procedure

Degrees of freedom

1. Label all the supports and joints (nodes)
2. Identify the DOFs
   - identify the known DOFs = 0
   - identify the unknown DOFs ≠ 0

Slope-deflection equations

- Summarize :

3 possible cases		
 	 (end –span w/ far-end pin/roller)	 (end –span w/ far-end free)

- Find **FEMs**
  - If a span is not loaded **FEMs = 0**
  - If a span is loaded then compute the **FEMs**
  - If FEM = **clockwise,** FEM = **+**

- Find Δ :
  - If no settlement Δ **= 0**
  - If settlement include Δ for the spans adjacent to the support that settles
    i. for each adjacent span: $\psi = \Delta / L$ ... radians

- Find slope-deflection equations:
  - For **internal spans** use slope-deflection **Eq.1** for **both ends**
  - For **end spans** with **far end fixed** use slope-deflection **Eq.1** for **both ends**
  - For **end spans** with **far end pin/roller** use slope-deflection **Eq.2** for **near end** only
  - For **end spans** with **far end free** do not write slope-deflection equations (instead use statics)

Eq #1: $$M_N = 2E\left(\frac{I}{L}\right)\left[2\theta_N + \theta_F - 3(\Psi)\right] + FEM_N$$

Eq #2: $$M_N = 3E\left(\frac{I}{L}\right)\left[\theta_N - (\Psi)\right] + FEM_N$$

Equilibrium equations

1. Write equilibrium equation at each **interior support** for **beams**
   (for frames these will be at the joints)
   - Draw the FBD of a small segment showing moments at the joint
   - Draw moments **counter-clockwise (+)** because of action-reaction principle

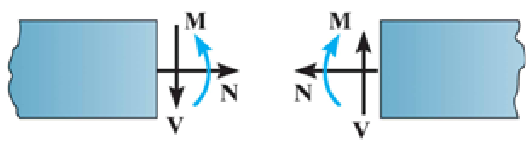
2. Generate an equation from each FBD by taking **∑M=0**

Solve for displacements and moments

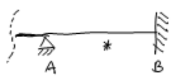
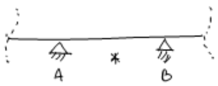
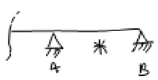
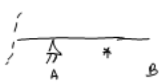
You should have ***x*** unknowns and ***x*** equations ...

1. Substitute slope-deflection relationships into the equilibrium equations and solve for the unknown displacements
2. Plug displacements into slope-deflection equations to get the internal moments at the ends of each member
   - If any of the results are **negative** they indicate **counter-clockwise** rotation; whereas **positive** results indicate **clockwise** rotation

3. Use the results to determine the moments and shears on isolated beam elements
4. Draw reactions on FBD of complete beam
5. Draw the shear and moment diagrams (in step 5 use our usual sign convention)



Guide:

Case		Which equation should you use ?	Equations	FEMs clockwise = + counter-clockwise = -
end-span fixed 	+ loads	Eq #1: $M_N = 2E\left(\frac{I}{L}\right)[2\theta_N + \theta_F - 3(\Psi)] + FEM_N$	M_{AB} M_{BA}	FEM_A FEM_B Use FEM table #1 (both ends fixed)
interior span 	+ no loads	Eq #1: $M_N = 2E\left(\frac{I}{L}\right)[2\theta_N + \theta_F - 3(\Psi)] + FEM_N$	M_{AB} M_{BA}	**FEM_A = 0** **FEM_B = 0**
End-span pin 	+ loads	Eq #2: $M_N = 3E\left(\frac{I}{L}\right)[\theta_N - (\Psi)] + FEM_N$	M_{AB} Where B is the pin/roller-end	FEM_A Use FEM table #2 (fixed - pin)
	+ no loads	Eq #2: $M_N = 3E\left(\frac{I}{L}\right)[\theta_N - (\Psi)] + FEM_N$	M_{AB} Where B is the pin/roller-end	**FEM_A = 0**
End-span free 	+ loads	Not any, instead use statics: Analyse as a cantilever, Moment = load *moment arm You'll need this moment for the joint "equilibrium" equations		
	+ no loads	Not any		

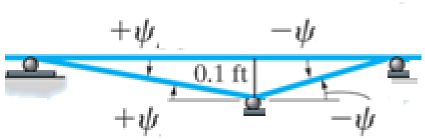
Relative linear displacements Δ

If **no** settlement , $\psi = \Delta/L = 0$

If have **downward** settlement include $\psi = \Delta/L$

If only "one" interior support settles ...

- if span is to the **left** of the support that settles, Δ = +
- if span is to the **right** of the support that settles, Δ = -



Otherwise need to think about rotations: ψ

- if ψ is to **clockwise**, Δ = +

Slope-Deflection for Frames

There are two classifications of planar fames that we are interested in for the Slope-Deflection Method:

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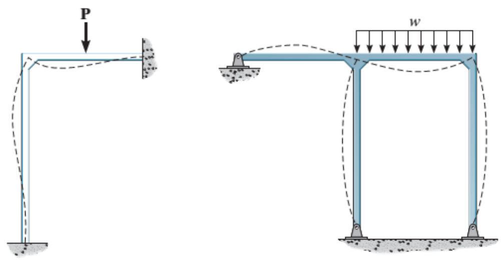
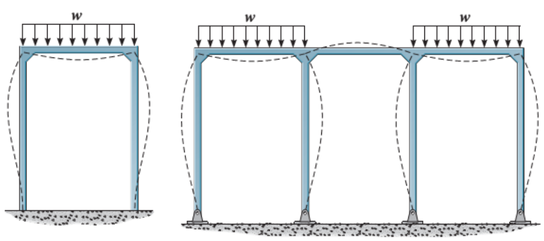


- braced against side sway



- symmetric structure
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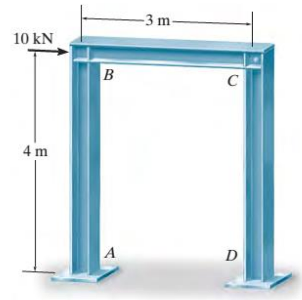
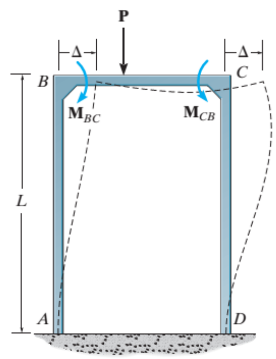


- unbraced with lateral load



- unbraced with unsymmetrical load or structure
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Non-Sway Frames:

- Translation of ***joint*** movement prevented
- Flexure of members between joints allowed
- Axial shortening of members is negligible

Sway Frames:

- Translation of ***joint*** occurs

- flexure of members between joints allowed
- Axial shortening of members is not negligible