Lecture 8.2: Moment distribution

Lecture outline:

1. Moment distribution - introduction
2. Methodology
3. Procedure
4. Stiffness modification factors

1. Moment Distribution

The moment distribution method uses similar principles as Slope-Deflection:

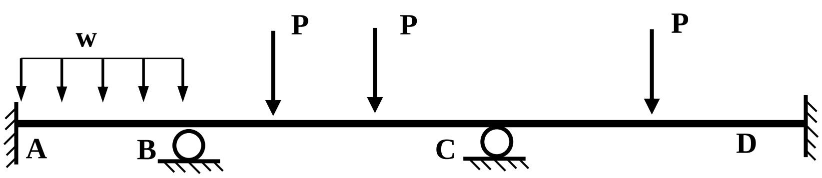
- Introducing compatibility conditions to indeterminate structures
- Building compatibility into equilibrium
- Ensuring equilibrium at all node locations (joints)

Unlike Slope-Deflection, moment distribution does not result in a system of simultaneous equations, rather, moment distribution uses an:

- Iterative procedure to enforce equilibrium at a joint and solve for moments.

The moment distribution system can be used to solve for moments in statically indeterminate:

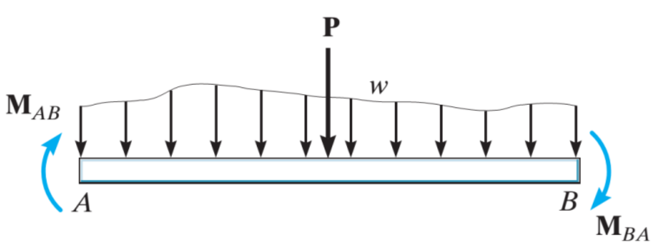
- Beams
- non-sway frames
- sway frames



2. Methodology

Sign Convention

We will use the same sign convention as that used in the Slope-Deflection method for consistency.



- Note: M_{AB} and M_{BA} are drawn as positive

Fixed End Moments

The fixed end moments used in the Moment Distribution method are the same as in the Slope-Deflection method (see Lecture 8.1).

Member Stiffness Factor

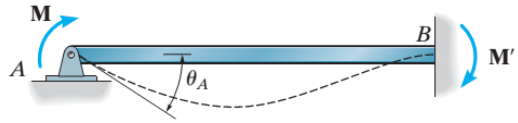
Recall that in building the Slope-Deflection moment equation for the moment at a joint we included:

- Rotation of the near joint, Θ_N
- Rotation of the far joint, Θ_F
- relative displacement between joints, Δ

We included these moment components directly in the equation for the moment at the joint. For the moment distribution method, we will use these moment components to facilitate the iterative solution process. We will do this by including the moment due to rotation in the calculation of the *Member Stiffness Factor*.

$$K = \frac{4EI}{L}$$

Far End Fixed



In the above, the member stiffness factor, K, is the amount of moment (at **A**) to rotate the above beam by at **A** by one radian.

Joint Stiffness Factor

If more than one member is framed into a joint (without any member releases), the joint stiffness factor is the total of the member stiffnesses at the joint.

$$K_T = \sum_{i=1}^{n} K_i$$

where:

- K_T is the *Joint Stiffness Factor*
- K_i is an individual *Member Stiffness Factor*
- n is the total number of members framing into a joint

Distribution Factor

The *Distribution Factor*, *DF*, is a factor that we use to perform the iteration to get moment equilibrium of the member. This factor allows us to use the relative stiffnesses of the members framing into the joint to divide up the moment proportionally.

$$DF_i = \frac{K_i}{K_T}$$

Carry-Over Factor

We use a carry-over factor of ½ when we distribute the moment.

3. Procedure

Moment Distribution Factor Procedure

To complete the moment-distribution process we successively lock and unlock joints in the structure and apply our carry-over and distribution factors. The best way to understand the process is through examples (as shown in the following sections), however, an attempt at describing the process is given below.

1. Label all joints in the structure
2. Compute the member stiffness factors for all sides of all joints
   - $K = \frac{4EI}{L}$
   - Members ***terminating*** in a pin (or roller) have
     - ***K = 0*** for the side of the joint without a member
   - Members ***terminating*** in a fixed end have
     - ***K*** **=** ∞ for the side of the joint without a member
3. Compute the Distribution Factor for all joints
   - $DF_i = \frac{K_i}{K_T}$
4. Set up a table with columns for each member grouped by joints and try to have connecting members side by side in the joint (see an example for this format)
5. Add the previously calculated stiffness factors and distribution factors to the table for each member
6. Compute the Fixed End Moment (FEM) for each member and add it to the table
7. Multiply the moment at each side of the joint by the distribution factor and apply it to each side of the joint with the ***opposite sign***
8. Carry over the moment from one side of the member to the other side using the moment carry-over factor (+ ½)
9. Repeat step 7 and 8 until the moments on each side of the joint converge to zero
10. Sum all the FEM, distributed and carried-over moments on a side of the joint to get the moment
11. Draw the moment distribution over the entire structure and find all reactions and other pertinent information.

EXAMPLE 12-1

Determine the internal moments at each support of the beam shown in Fig. 12–7*a*. *EI* is constant.

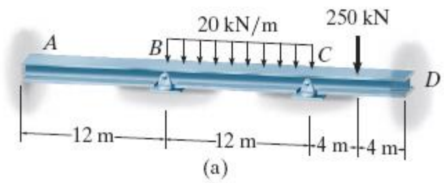


Fig. 12–7

Solution

The distribution factors at each joint must be computed first.* The stiffness factors for the members are

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$\text{DF}_{AB} = \text{DF}_{DC} = 0 \qquad \text{DF}_{BA} = \text{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\text{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad \text{DF}_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$

EXAMPLE 12-1 - (Continued)

The fixed-end moments are

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250 \text{ kN}\cdot\text{m} \qquad (\text{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN}\cdot\text{m}$$

Starting with the FEMs, line 4, Fig. 12–7*b*, the moments at joints *B* and *C* are distributed *simultaneously*, line 5. These moments are then carried over *simultaneously* to the respective ends of each span, line 6. The resulting moments are again simultaneously distributed and carried over, lines 7 and 8. The process is continued until the resulting moments are diminished an appropriate amount, line 13. The resulting moments are found by summation, line 14.

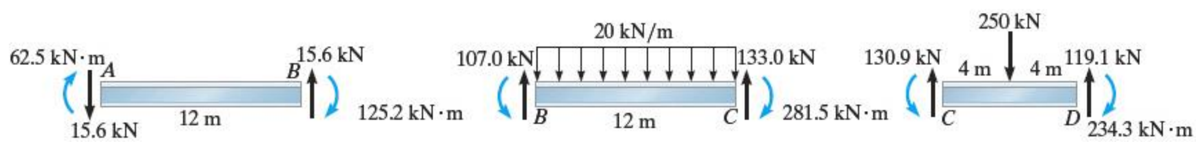
Placing the moments on each beam span and applying the equations of equilibrium yields the end shears shown in Fig. 12–7*c* and the bending-moment diagram for the entire beam, Fig. 12–7*d*.

*Here we have used the stiffness factor $4EI/L$; however, the relative stiffness factor I/L could also have been used.

EXAMPLE 12-1 (Continued)

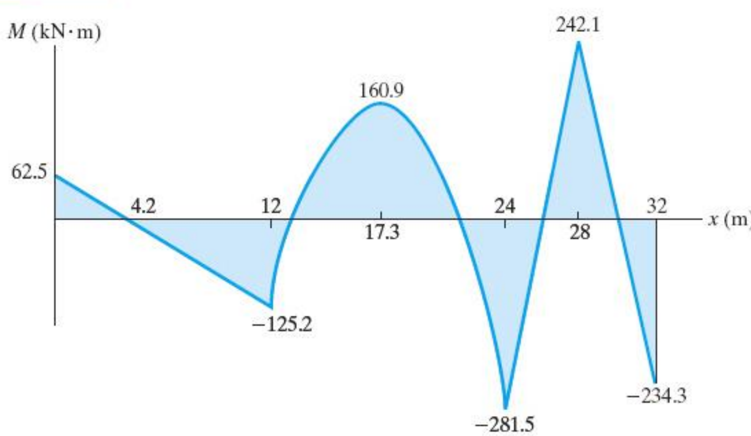
Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

(b)



(c)

EXAMPLE 12-1 (Continued)



(d)

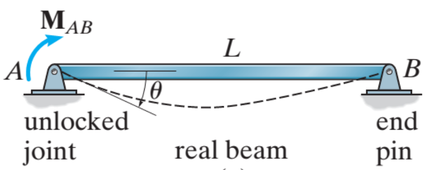
4. Stiffness Modification Factors

The moment distribution method can take considerable effort to solve if there are non-fixed ends. We can modify our member stiffness factor to represent the case of a pin or roller ended member. Further modifications can be made to account for structural symmetry and anti-symmetry.

Member pin-ended at far end

In order to solve for the member stiffness factor for pin-ended members, we will need to:

- solve for the amount of moment needed to induce 1 radian of rotation for a pin-ended beam.



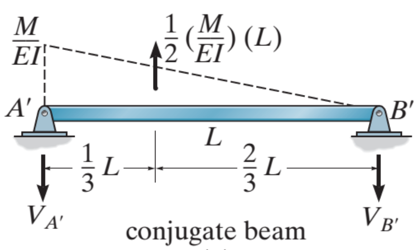
(a)

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

or

$$M = \frac{3EI}{L}\theta$$



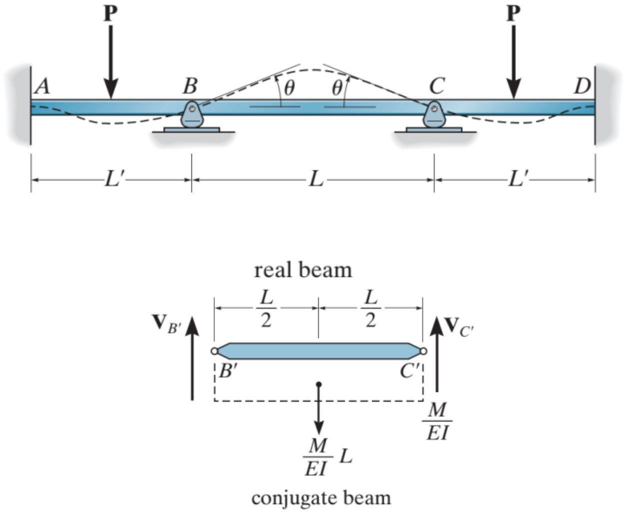
(b)

Far End Pinned

$$K = \frac{3EI}{L}$$

Structural Symmetry

We can also simply problems by recognizing symmetry in a structure.



$$-V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

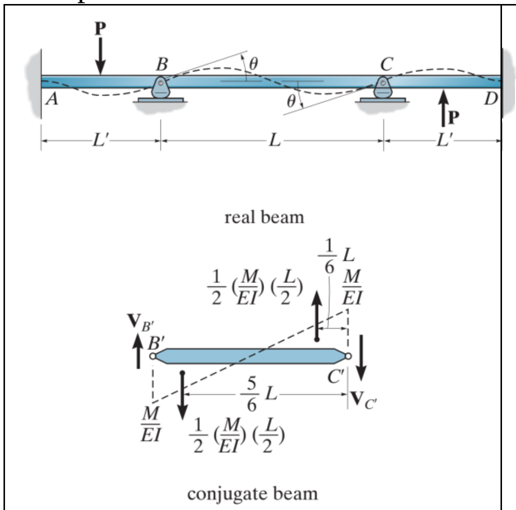
$$M = \frac{2EI}{L}\theta$$

$$\boxed{K = \frac{2EI}{L}}$$

<u>Note</u>: When we use symmetry in the moment distribution method, we do not apply the carry-overs across the line of symmetry (if we have used the modified stiffness factor).

Anti-Symmetry

The procedure for structures with anti-symmetry is similar to that of those with symmetry except a different member stiffness factor is used.



$$\circlearrowleft + \Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

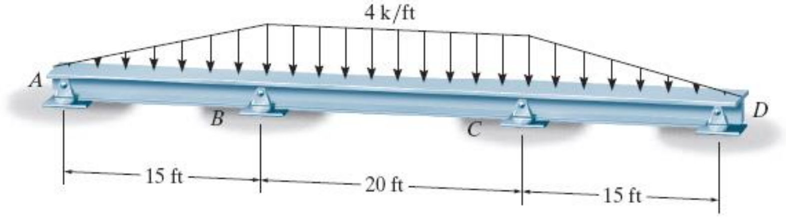
or

$$M = \frac{6EI}{L}\theta$$

$$\boxed{K = \frac{6EI}{L}}$$

EXAMPLE 12-3

Determine the internal moments at the supports for the beam shown in Fig. 12–13*a*. *EI* is constant.



(a)

Fig. 12–13

Solution

By inspection, the beam and loading are symmetrical. Thus, we will apply $K = 2EI/L$ to compute the stiffness factor of the center span BC and therefore use only the left half of the beam for the analysis. The analysis can be shortened even further by using $K = 3EI/L$ for computing the stiffness factor of segment AB since the far end A is pinned. Furthermore, the distribution of moment at A can be skipped by using the FEM for a triangular loading on a span with one end fixed and the other pinned. Thus,

EXAMPLE 12-3 *(Continued)*

$$K_{AB} = \frac{3EI}{15} \qquad \text{(using Eq. 12–4)}$$

$$K_{BC} = \frac{2EI}{20} \qquad \text{(using Eq. 12–5)}$$

$$\text{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\text{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\text{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\text{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3\ \text{k}\cdot\text{ft}$$

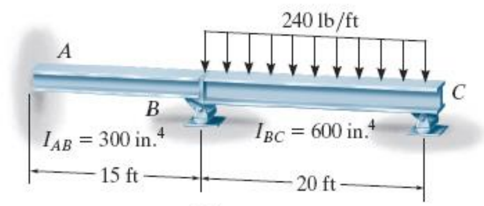
These data are listed in the table in Fig. 12–13*b*. Computing the stiffness factors as shown above considerably reduces the analysis, since only joint B must be balanced and carry-overs to joints A and C are not necessary. Obviously, joint C is subjected to the same internal moment of 108.9 k · ft.

Joint	A	B	
Member	AB	BA	BC
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

(b)

EXAMPLE 12-4

Determine the internal moments at the supports of the beam shown in Fig. 12–14*a*. The moment of inertia of the two spans is shown in the figure.



(a)

Fig. 12–14

Solution

Since the beam is roller supported at its far end C, the stiffness of span BC will be computed on the basis of $K = 3EI/L$. We have

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

EXAMPLE 12-4 - *(Continued)*

Thus,

$$\mathrm{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\mathrm{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\mathrm{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\mathrm{DF}_{CB} = \frac{90E}{90E} = 1$$

Further simplification of the distribution method for this problem is possible by realizing that a *single* fixed-end moment for the end span BC can be used. Using the right-hand column of the table on the inside back cover for a uniformly loaded span having one side fixed, the other pinned, we have

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000 \text{ lb} \cdot \text{ft}$$

EXAMPLE 12-4 (Continued)

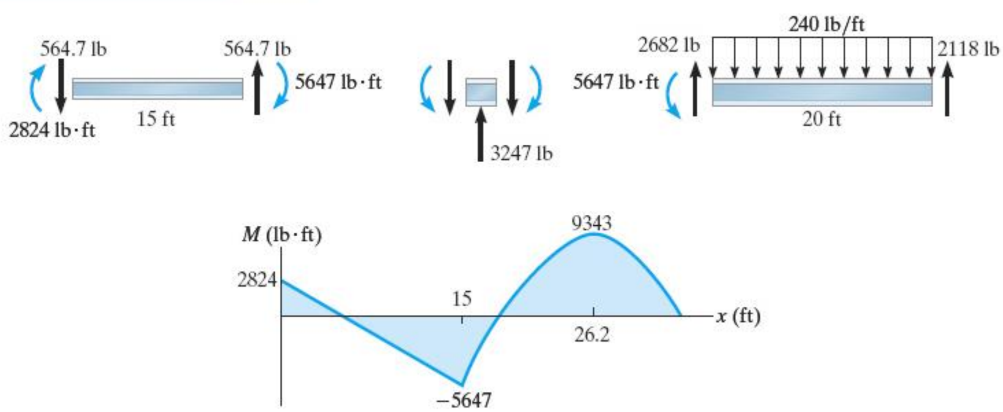
The foregoing data are entered into the table in Fig. 12–14*b* and the moment distribution is carried out. By comparison with Fig. 12–6*b*, this method considerably simplifies the distribution.

Using the results, the beam's end shears and moment diagrams are shown in Fig. 12–14*c*.

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4706	0.5294	1
FEM			−12 000	
Dist.		5647.2	6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0

(b)

EXAMPLE 12-4 (Continued)



(c)