MATH3705B — Test 4: 14:35–15:25, March 21

Surname ______________ **First Name** ______________ **Student #** __________

Total: 15 points. No partial marks for Questions 1-2.
Closed book! Non-programmer calculators are allowed!

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1. (2 points) The solution of the wave equation $u_{tt} = c^2 u_{xx}$, $0 < x < L, t > 0$, subject to $u(0,t) = 0, u(L,t) = 0$, $u(x,0) = f(x), u_t(x,0) = g(x)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi ct}{L}) + b_n \sin(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where $b_n =$

(a) $\frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$ (b) $\frac{2}{n\pi c} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$ (c) $\frac{1}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$

(d) $\frac{1}{n\pi c} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$ (e) $\frac{2}{n\pi c} \int_0^L g(x) \cos\left(\frac{n\pi x}{L}\right) dx.$

Solution: (a).

2. (2 points) Let $u(x,y)$ be the polynomial solution of $u_{xx} + u_{yy} = 0$, $0 < x < 1, 0 < y < 1$, subject to $u(x,0) = x - 1, u(x,1) = 4x - 1, u(0,y) = -1, u(1,y) = 3y$. Find $u(0.1, 0.1)$.

(a) -0.73 (b) -1.3 (c) -0.3 (d) -0.15 (e) -0.87

Solution: (e).

Let $u(x,y) = ax + by + cxy + d$.

By $u(x,0) = x - 1 = ax + d$, we imply that $a = 1, d = -1 \Rightarrow u(x,y) = x + by + cxy - 1$;

By $u(0,y) = -1 = by - 1, \Rightarrow b = 0, \Rightarrow u(x,y) = x + cxy - 1$;

By $u(1,y) = 3y = 1 + cy, \Rightarrow c = 3, \Rightarrow$

$u(x,y) = x + 3xy - 1. \Rightarrow u(0.1, 0.1) = 0.1 + 0.03 - 1 = -0.87$

3. (2 points) Use "separation of variables" to write the PDE $u_{tt} = c^2 u_{xx}$ as several ODEs (do not solve them).

Solution: Let $u(x,t) = X(x)T(t)$ and substitute it into the equation $u_{tt} = c^2 u_{xx}$, to obtain:

$$X(x)T''(t) = c^2 X''(x)T(t), \Rightarrow$$
$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = constant\ say,\ \lambda.$$

Thus

$$X''(x) - \lambda X(x) = 0,$$
$$T''(t) - \lambda c^2 T(t) = 0.$$

4. (4 points) The bounded solution of the Laplace equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 2$ is given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Given the boundary condition $u(2,\theta) = 2\cos^2(\theta) + \sin(3\theta)$, find a_0, a_n and b_n.

Solution: The solution of the PDE above is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Note that $2\cos^2(\theta) = 1 + \cos(2\theta)$. Thus

$$u(2,\theta) = 1 + \cos(2\theta) + \sin(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

which implies that

$a_0 = 2, 1 = 2^{-2}a_2, a_n = 0$ for $n \neq 0, 2$;

1 point

$1 = 2^{-3}b_3$, $b_n = 0$ for $n \neq 3$.

1 point

Thus,

$a_0 = 2, a_2 = 4, a_n = 0$ for $n \neq 0, 2$;

1 point

$b_3 = 8$, $b_n = 0$ for $n \neq 3$.

1 point

5. (5 points) The solution of the wave equation $u_{tt} = 4u_{xx}$, $0 < x < 2$, $t > 0$ subject to $u(0,t) = u(2,t) = 0$, $u(x,0) = 3\sin(2\pi x)$, $u_t(x,0) = 16\sin(4\pi x)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)] \sin(\frac{n\pi x}{2}).$$

Find all a_n and b_n, and the detail solution $u(x,t)$.

Solution: From

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(n\pi t) + b_n \sin(n\pi t)] \sin(\frac{n\pi x}{2})$$

we imply that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{2}) = 3\sin(2\pi x) \Rightarrow$$

$$a_4 = 3, \quad a_n = 0 \ \ (n \neq 4).$$

2 points

Note that

$$u_t(x,t) = \sum_{n=1}^{\infty} [-n\pi a_n \sin(n\pi t) + n\pi b_n \cos(n\pi t)] \sin(\frac{n\pi x}{2}). \Rightarrow$$

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi b_n \sin(\frac{n\pi x}{2}) = 3\sin(4\pi x). \Rightarrow$$

$$8\pi b_8 = 16, \quad b_n = 0 \ \ (n \neq 8). \Rightarrow b_8 = \frac{2}{\pi}.$$

2 points

Thus

$$u(x,t) = 3\cos(4\pi t)\sin(2\pi x) + \frac{2}{\pi}\sin(8\pi t)\sin(4\pi x).$$

1 point