Problem#1

Classify the following differential equation by order and linearity.

a) $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^4$

b) $y'' = \sqrt{5x^6 + (\sin x)y^{(8)}}$

c) $e^{-x}y' + (4\sin x)y = (\tan x)y'' - 4e^{-x^2}$

d) $\frac{dy}{dt} = 1 - t^2$

e) $x^2y'' + xy' + (x^2 - v^2)y = 0$

Solution:

	Order	**Linearity**
a)	2	non linear
b)	8	non linear
c)	2	linear
d)	1	linear
e)	2	linear

Problem#2:

Show that $y = \frac{1}{4}e^{3x} + ce^{-x}$ is a solution to the DE $\frac{dy}{dx} + y = e^{3x}$. What is the largest interval of existence for a solution to the IVP *y (-3)=15?*

Solution:

Solution # 1

$$y = \frac{1}{4} e^{3x} + c e^{-x}$$

$$y' = \frac{3}{4} e^{3x} - c e^{-x}$$

$$\frac{dy}{dx} + y = \frac{3}{4} e^{3x} - c e^{-x} + \frac{1}{4} e^{3x} + c e^{-x} = \frac{4}{4} e^{3x} = e^{3x}$$

L.H.S. = R.H.S.

This is a solution.

Solution # 2

$$\frac{dy}{dx} + y = e^{3x} \qquad P(x) = 1 \qquad \int P(x)\,dx = x$$

integrating factor $\mu = e^{x}$

$$\int (e^{x} y)'\, dy = \int e^{4x}\, dx$$

$$e^{x} y = \frac{1}{4} e^{4x} + c \Rightarrow y = \frac{1}{4} e^{3x} + c e^{-x}$$

the largest interval because valued e is positive

$x \in \,]-\infty, \infty[$

Problem#3

Solve the differential equation $y' = y^2 - 4y + 3$ given initial condition $y(0) = 3$.

Solution:

$$y' = y^2 - 4y + 3$$

$$\frac{dy}{dx} = y^2 - 4y + 3$$

$$\frac{A}{(y-3)} + \frac{B}{(y-1)} = \frac{1}{(y-1)(y-3)}$$

$$A(y-1) + B(y-3) = 1$$

$$(A+B)y = 0 \Rightarrow A = -B$$

$$-A - 3B = 1$$

$$B - 3B = 1 \Rightarrow B = -\frac{1}{2}$$

$$A = \frac{1}{2}$$

$$= \int \frac{dy}{y^2 - 4y + 3} = \int dx$$

$$= \int \frac{dy}{(y-3)(y-1)} = \int dx$$

$$= \frac{1}{2} \int \frac{1}{y-3} - \frac{1}{y-1}\, dy = \int dx$$

$$= \frac{1}{2} \left[\ln(y-3) - \ln(y-1) \right] = x + C$$

$$= \frac{1}{2} \ln\left(\frac{y-3}{y-1}\right) = x + C \Rightarrow \ln\left(\frac{y-3}{y-1}\right) = 2x + C$$

$$\frac{y-3}{y-1} = C e^{2x} \Rightarrow$$

$$x = 0 \Rightarrow y = 3$$

$$0 = C e^{0} \Rightarrow C = 0$$

$$\frac{1}{e^{2x}} \left(\frac{y-3}{y-1}\right) = 0$$

Problem#4

Solve the initial value problem $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

Solution

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)} \quad , \; y(0) = 2$$

$$dy\,(y - yx^2) = (xy^2 - \cos x \sin x)\,dx$$

$$(xy^2 - \cos x \sin x)\,dx + (yx^2 - y)\,dy = 0$$

$$M = xy^2 - \cos x \sin x \qquad \frac{\partial M}{\partial y} = 2xy$$

$$= \text{Equal}$$

$$N = yx^2 - y \qquad \frac{\partial N}{\partial x} = 2xy$$

Exact Equation.

$$f(x,y) = \int M\,dx = \int N\,dy$$

$$= \int yx^2 - y \;dy = \frac{1}{2}x^2y^2 - \frac{y^2}{2} + g(x)$$

$$\frac{\partial f}{\partial x} = M = xy^2 - \cos x \sin x = xy^2 + g'(x)$$

$$g'(x) = -\cos x \sin x \implies g(x) = -\frac{1}{2}\sin^2 x \quad \text{or} \quad \frac{1}{2}\cos^2 x$$

$$f(x,y) = \frac{1}{2}x^2y^2 - \frac{y^2}{2} - \frac{1}{2}\sin^2 x = C_1$$

OR

$$= \frac{1}{2}x^2y^2 - \frac{y^2}{2} + \frac{1}{2}\cos^2 x = C_2$$

For $x = 0 \implies y = 2$

$$0 - 2 - \frac{1}{2}(0) \implies = C_1 \implies C_1 = -2$$

OR

$$0 - 2 + \frac{1}{2}(1) = C_2 \implies C_2 = -\frac{3}{2}$$

Problem#4

Solve the initial value problem $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

Solution

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$$

$$\frac{dy}{dx}(y)(1-x^2) = xy^2 - \cos x \sin x$$

let $u = y^2$

$du = 2y\,dy$

$dy = \frac{1}{2y}\,du$

$$(1-x^2)\frac{dy}{dx} = xy - (\cos x \sin x)y^{-1}$$

$$(1-x^2)\frac{dy}{dx} - xy = -((\cos x \sin x)y^{-1}$$

$$y(1-x^2)\frac{dy}{dx} - xy^2 = -(\cos x \sin x)$$

$$\frac{1}{2y} y(1-x^2)\frac{du}{dx} - xu = -(\cos x \sin x)$$

$$\frac{du}{dx} - \frac{2x}{1-x^2}u = \frac{-2(\cos x \sin x)}{1-x^2}$$

$$P(x) = \frac{-2x}{1-x^2} \qquad \int P(x)\,dx = \ln(1-x^2)$$

integrating factor $\Rightarrow e^{\ln|(1-x^2)|} = 1-x^2$

$$\int ((1-x^2)u)'\,du = \int \frac{-2(\cos x \sin x)}{(1-x^2)}(1-x^2)\,dx$$

$$(1-x^2)u = -\sin^2 x + C_1 \quad \text{OR} \quad \cos^2 x + C_2$$

$$y^2 = \frac{-\sin^2 x + C_1}{1-x^2}$$

$$y = \pm\sqrt{\frac{-\sin^2 x + C_1}{1-x^2}} \quad \text{OR} \quad \pm\sqrt{\frac{\cos^2 x + C_2}{1-x^2}}$$

$y(0) = 2$

$$4 = \frac{-0 + c_1}{1} \Rightarrow c_1 = 4$$

OR $$4 = \frac{1 + c_2}{1} \Rightarrow c_2 = 3$$

Problem#5

Solve the differential equation $\frac{dx}{dy} = x(yx^3 - 1)$

Solution:

$$\frac{dx}{dy} = yx^4 - x \quad \Rightarrow \quad \frac{dx}{dy} + x = yx^4$$

$$\frac{1}{x^4}\frac{dx}{dy} + \frac{1}{x^3} = y \qquad \text{let } u = \frac{1}{x^3} = x^{-3}$$

$$\frac{du}{dx} = -3x^{-4}$$

$$-3x^{-4}dx = du$$

$$dx = -\frac{1}{3}x^4\,du$$

$$-\frac{1}{3}\frac{du}{dy} + u = y$$

$$+\frac{du}{dy} - 3u = -3y$$

$$P(y) = -3 \Rightarrow \int P(y)\,dy = \int -3\,dy = -3y$$

integrating factor e^{-3y}

$$v = y$$

$$dm = -3e^{-3y}$$

$$dv = dy$$

$$m = e^{-3}$$

$$\int (e^{-3y}u)'\,du = \int -3y\,e^{-3y}\,dy$$

$$e^{-3y}u = ye^{-3y} - \int e^{-3y}\,dy$$

$$e^{-3y}u = ye^{-3y} + \frac{1}{3}e^{-3y} + C$$

$$u = y + \frac{1}{3} + Ce^{3y}$$

$$\frac{1}{x^3} = y + \frac{1}{3} + Ce^{3y}$$

$$x^3 = \frac{1}{y + \frac{1}{3} + Ce^{3y}}$$

$$x = \left(\frac{1}{y + \frac{1}{3} + Ce^{3y}}\right)^{1/3}$$