Mid-term Examination – Fall 2017

Course title: **Applied Probability**
Date of examination: **30/10/2017, 1:15 PM**
Instructor's name: **Dr. Farhad Shokoohi**
Number of pages including the cover page: **6**
Student's name: ____________________

Class: **MAST 221**
Time duration: **75 minutes**
Total marks: **35**
Number of questions: **6**
Student's ID: ____________________

Instructions:

I) This is a closed book exam.

II) Solve all the questions.

III) Only faculty approved calculators are allowed.

IV) Organize your work, in a reasonably neat and coherent way, in the booklet.

V) Work scattered all over the page without a clear ordering will receive very little credit.

VI) Mysterious or unsupported answers will not receive full credit.

VII) A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit.

VIII) An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

IX) Write your name and your student ID on the first page of booklet, and your initials on each of the booklet page that is used to answer questions.

Q.1) Provide the definition for the following terms.

(a) Sample Space; 1 mark

Solution: The set of all possible outcomes of an experiment is called the **sample space** and os is usually denoted by the letter S. Each outcome in a sample space is called an **element** of the sample space, or simply a **sample point**.

(b) Event; 1 mark

Solution: An **event** is a subset of a sample space.

(c) Mutually Exclusive Events; 1 mark

Solution: Two events having no elements in common are said to be **mutually exclusive**.

(d) Independence of Two Events; 1 mark

Solution: Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B).$$

(e) Conditional Probability of Two Events; 1 mark

Solution: If A and B are any two events in a sample space S and $P(A) \neq 0$, the **conditional probability** of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

(f) Moment Generating Function for a Continuous Random Variable. 1 mark

Solution: The **moment generating function** of a continuous random variable X, where it exists, is given by

$$M_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx.$$

Q.2) In a certain community, 10 percent of all adults have diabetes. If a health service in this community correctly diagnoses 80 percent of all persons with diabetes as having the disease and incorrectly diagnoses 5 percent of all persons without diabetes as having the disease, find the probabilities that

(a) the community health service will diagnose an adult as having diabetes; 3 marks

Solution: Assume "adults with diabetes" are denoted by A, and "diagnosing diabetes" are denoted by B. Therefore, based on the information provided we have

$$P(A) = 0.1 \quad , \quad P(B|A) = 0.8 \quad \text{and} \quad P(B|A^c) = 0.05.$$

Hence, using the rule of total probability we get

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 0.8 \times 0.1 + 0.05 \times 0.9 = \frac{1}{8} = 0.125.$$

(b) an adult diagnosed by the health service as having diabetes actually has the disease. 3 marks

Solution: We need to find $P(A|B)$. Using the Bayes' rule we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.8 \times 0.1}{0.125} = 0.64$$

Q.3) (a) Determine c so that the function $f(y) = c\ y\ e^{-y^2}$, for $y > 0$, can serve as the probability density function of a random variable. 2 marks

Solution: This is a continuous random variable; therefore, we should have

- $0 \leq f(x)$, and
- $\int_{-\infty}^{\infty} f(x)dx = 1.$

From the first part we see that $0 \leq c$; and from the second part we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_0^{\infty} c\ y\ e^{-y^2} dx = \frac{-\,c}{2}\ e^{-y^2}\Big]_0^{\infty} = \lim_{y\to\infty}\left(\frac{-\,c}{2}\ e^{-y^2}\right) - \left(\frac{-\,c}{2}\ e^0\right) \\ &= 0\ +\ \frac{c}{2} = 1\end{aligned}$$

which leads to

$$c = 2.$$

(b) Find $F(y)$. 2 marks

Solution: The cumulative distribution function, by definition, is

$$F(y) = P(Y \leq y) = \int_{-\infty}^{y} f(t)dt = \int_0^{y} 2\ t\ e^{-t^2}\ dt = -e^{-t^2}\Big]_0^{y} = 1 - e^{-y^2};$$

hence,

$$F(y) = \begin{cases} 0 & \text{for} \quad y < 0 \\ 1 - e^{-y^2} & \text{for} \quad 0 \leq y \end{cases}$$

Question 3 continues on the next page...

(c) Calculate $P(1 < Y < 3)$ based on the formula derived in Q.3(b). 2 marks

Solution: Since Y is a continuous random variable, we have

$$\begin{aligned} P(1 < Y < 3) &= P(1 < Y \le 3) = P(Y \le 3) - P(Y \le 1) = F(3) - F(1) \\ &= 1 - e^{-3^2} - \left(1 - e^{-1^2}\right) = e^{-1} - e^{-9} \\ &= 0.3677560314 \end{aligned}$$

Q.4) Assume the joint probability density of X and Y is given by

$$f(x, y) = \begin{cases} 2x(x - y), & \text{for } 0 < x < 1, \ -x < y < x \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the marginal density of X. 2 marks

Solution: For the marginal pdf of X we have

$$\begin{aligned} g(x) &= \int_{-x}^{x} 2x(x - y)dy = 2x \left(xy - \frac{y^2}{2}\right)\Bigg]_{-x}^{x} = 2x^2y - xy^2\Big]_{-x}^{x} \\ &= \left(2x^2x - xx^2\right) - \left(2x^2(-x) - x(-x)^2\right) \\ &= 2x^3 - x^3 + 2x^3 + x^3 \\ &= 4x^3. \end{aligned}$$

Hence,

$$g(x) = \begin{cases} 4x^3 & \text{for } \ 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) Find the variance of X. 4 marks

Solution: First, we need to find the mean of X; we have

$$E[X] = \int_0^1 4xx^3 dx = \frac{4}{5}x^5\Big]_0^1 = \frac{4}{5}.$$

Then we have,

$$E[X^2] = \int_0^1 4x^2x^3 dx = \frac{4}{6}x^6\Big]_0^1 = \frac{4}{6} = \frac{2}{3}.$$

Finally,

$$Var[X] = E[X^2] - \Big(E[X]\Big)^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}.$$

Q.5) If the joint probability density of X, Y and Z is given by

$$f(x,y,z) = \begin{cases} \frac{1}{3}\left(2\,x + 3\,y + z\right) & \text{for } 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) find $P\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right)$. 1 mark

Solution: Since X, Y and Z are continuous random variables the probability of a point in the sample space is zero, i.e.,

$$P\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right) = 0.$$

(b) find the expected value of $W = X^2 - YZ$. 5 marks

Solution: We have,

$$\begin{aligned}
E[W] &= \int_0^1\int_0^1\int_0^1 (x^2 - yz)\,\frac{1}{3}\left(2\,x + 3\,y + z\right)\,dz\,dy\,dx \\
&= \frac{1}{3}\int_0^1\int_0^1\int_0^1 \left(2\,x^3 + 3\,y\,x^2 + z\,x^2 - 2\,x\,y\,z - 3\,y^2\,z - y\,z^2\right)\,dz\,dy\,dx \\
&= \frac{1}{3}\int_0^1\int_0^1 \left(2\,x^3 z + 3\,yx^2 z + \frac{1}{2}\,z^2x^2 - xyz^2 - \frac{3}{2}\,y^2z^2 - \frac{1}{3}\,yz^3\right)\Bigg]_0^1\,dy\,dx \\
&= \frac{1}{3}\int_0^1\int_0^1 \left(2\,x^3 + 3\,yx^2 + \frac{1}{2}\,x^2 - xy - \frac{3}{2}\,y^2 - \frac{1}{3}\,y\right)\,dy\,dx \\
&= \frac{1}{3}\int_0^1 \left(2\,x^3y + \frac{3}{2}\,y^2x^2 + \frac{1}{2}\,x^2y - \frac{1}{2}\,xy^2 - \frac{1}{2}\,y^3 - \frac{1}{6}\,y^2\right)\Bigg]_0^1\,dx \\
&= \frac{1}{3}\int_0^1 \left(2\,x^3 + \frac{3}{2}\,x^2 + \frac{1}{2}\,x^2 - \frac{1}{2}\,x - \frac{1}{2} - \frac{1}{6}\right)\,dx \\
&= \frac{1}{3}\int_0^1 \left(2\,x^3 + 2\,x^2 - \frac{1}{2}\,x - \frac{2}{3}\right)\,dx \\
&= \frac{1}{3}\left(\frac{1}{2}\,x^4 + \frac{2}{3}\,x^3 - \frac{1}{4}\,x^2 - \frac{2}{3}\,x\right)\Bigg]_0^1 \\
&= \frac{1}{3}\left(\frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{2}{3}\right) \\
&= \frac{1}{12}
\end{aligned}$$

Q.6) Prove by induction that

$$P\left(E_1 \cup E_2 \cup \cdots \cup E_n\right) \leq \sum\nolimits_{i=1}^{n} P(E_i),$$

for any finite sequence of events $E_1, E_2, \ldots, E_n$.

Solution: To prove using induction three steps should be carried out.

Step1: We show that the inequality holds for $n = 2$. This is done by using a theorem that the probability of the union of two events is equal to the probability of each event minus the probability of their interaction, that is,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

Since $0 \leq P(E_1 \cap E_2)$, then

$$P(E_1 \cup E_2) \leq P(E_1) + P(E_2).$$

Step 2: In this step we assume that the inequality holds for $n = k$ events; that is,

$$P\left(E_1 \cup E_2 \cup \cdots \cup E_k\right) \leq \sum\nolimits_{i=1}^{k} P(E_i),$$

Step 3: Now we prove that the inequality holds for $n = k + 1$, that is,

$$\begin{aligned} P\left(E_1 \cup E_2 \cup \cdots \cup E_{k+1}\right) &= P\left(\left\{E_1 \cup E_2 \cup \cdots \cup E_k\right\} \cup E_{k+1}\right) \\ &= P\left(\left\{E_1 \cup E_2 \cup \cdots \cup E_k\right\}\right) + P(E_{k+1}) \\ &\quad -P\left(\left\{E_1 \cup E_2 \cup \cdots \cup E_k\right\} \cap E_{k+1}\right) \\ &\leq P\left(\left\{E_1 \cup E_2 \cup \cdots \cup E_k\right\}\right) + P(E_{k+1}) \\ &\leq \sum\nolimits_{i=1}^{k} P(E_i) + P(E_{k+1}) \\ &= \sum\nolimits_{i=1}^{k+1} P(E_i). \end{aligned}$$

Therefore, this inequality holds for any finite number n.

5 marks