Assignment 4 - Solutions

20th November, 2017

5.2 The Moment Generating Function (MGF) of the Uniform distribution is given as

$$M_X(t) = E[e^{tX}] = \sum_{x=1}^{k} e^{tx}\frac{1}{k} = \frac{1}{k}\sum_{x=1}^{k} e^{tx} = \frac{1}{k}\,e^{t}\sum_{x=0}^{k-1} e^{tx}$$

Recall the geometric series

$$S_n = \sum_{i=0}^{n} r^n = \frac{1-r^{n+1}}{1-r} \quad \text{for} \quad |r| < -1$$

hence

$$M_X(t) = \frac{1}{k}\,e^{t}\frac{1-(e^t)^{k-1+1}}{1-e^t} = \frac{1}{k}\,e^{t}\frac{1-e^{kt}}{1-e^t} = \frac{e^t - e^{(k+1)t}}{k(1-e^t)}$$

Note that in this MGF we must have $-\infty < t < 0$.

To obtain mean we can use $\mu_X = M'_X\Big|_{t=0}$; However, the nominator and denominator become zero if we evaluate them at zero thus we use the limit and Hospital's Rule as follows:

$$\begin{aligned}
\mu &= \lim_{t\to 0} M'_X(t) = \lim_{t\to 0} \frac{\left[e^t - (k+1)e^{(k+1)t}\right]\left[k(1-e^t)\right] + ke^t\left[e^t - e^{(k+1)t}\right]}{k^2(1-e^t)^2} \\
&= \lim_{t\to 0} \frac{e^t - (k+1)e^{(k+1)t} + ke^{(k+2)t}}{k(1-e^t)^2} && \text{[simplifying]} \\
&= \lim_{t\to 0} \frac{e^t - (k+1)^2e^{(k+1)t} + k(k+2)e^{(k+2)t}}{-2ke^t(1-e^t)} && \text{[Hospital's Rule]} \\
&= \lim_{t\to 0} \frac{e^t - (k+1)^3e^{(k+1)t} + k(k+2)^2e^{(k+2)t}}{-2ke^t(1-e^t) + 2ke^te^t} && \text{[Hospital's Rule]} \\
&= \frac{1-(k+1)^3 + k(k+2)^2}{0+2k} && \text{[applying limit]} \\
&= \frac{k+1}{2} && \text{[simplifying]}
\end{aligned}$$

The result is the same as that of Exercise 5.1.

5.4 First note that $\alpha_3 = \dfrac{\mu_3}{\sigma^3}$ and $\alpha_4 = \dfrac{\mu_4}{\sigma^4}$ based on the exercises 4.26 and 4.27, respectively. Also for Bernoulli distribution we have

$$\mu = \mu_1' = \theta, \quad \mu_2' = \theta, \quad \mu_3' = \theta \quad \mu_4' = \theta, \quad \sigma^2 = \theta(1-\theta)$$

Using what we learnt in Exercise 4.25 we have

$$\mu_3 = \mu_3^{'} - 3\mu\mu_2^{'} + 2\mu^3$$

$$\mu_4 = \mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4$$

(a) For skewness we have

$$\mu_3 = \theta - 3\theta\theta + 2\theta^3 = \theta(1 - 3\theta + 2\theta^2) = \theta(1-2\theta)(1-\theta)$$

Therefore, we have

$$\alpha_3 = \frac{\theta(1-\theta)(1-2\theta)}{\theta(1-\theta)\sqrt{\theta(1-\theta)}} = \frac{1-2\theta}{\sqrt{\theta(1-\theta)}}$$

(b) For peakedness we have

$$\mu_4 = \theta - 4\theta^2 + 6\theta^3 - 3\theta^4 = \theta(1 - 4\theta + 6\theta^2 - 3\theta^2) = \theta(1-\theta)[1 - 3\theta(1-\theta)]$$

hence,

$$\alpha_4 = \frac{\theta(1-\theta)[1-3\theta(1-\theta)]}{\theta^2(1-\theta)^2} = \frac{1-3\theta(1-\theta)}{\theta(1-\theta)}$$

5.12 Based on Exercise 5.11, the rth factorial moment is defined as

$$\mu_{(r)}' = E[X(X-1)\cdots(X-r+1)]$$

We now verify that the rth derivative of factorial moment-generating function of a discrete random variable produces the $r$the factorial moment. We try for different values of r as follows:

$$F'(x) = \sum xt^{t-1}f(x) \quad \Rightarrow \quad F'(t)\Big|_{t=1} = \sum xf(x) = E[X] = \mu_{(1)}'$$

$$F''(x) = \sum x(x-1)t^{t-2}f(x) \quad \Rightarrow \quad F''(t)\Big|_{t=1} = \sum x(x-1)f(x) = E[X(X-1)] = \mu_{(2)}'$$

$$F'''(x) = \sum x(x-1)(x-1)t^{t-3}f(x) \Rightarrow F'''(t)\Big|_{t=1} = \sum x(x-1)(x-2)f(x) = E[X(X-1)(X-2)] = \mu_{(3)}'$$

and so on.

5.20 The Geometric distribution is given by

$$g(x;\theta) = \theta(1-\theta)^{x-1} \quad x = 1, 2, 3, \ldots$$

We then have

$$\begin{aligned}
M_X(t) &= \sum_{x=1}^{\infty} e^{tx}\theta(1-\theta)^{x-1} \\
&= \sum_{x=1}^{\infty} \theta \frac{[e^t(1-\theta)]^x}{1-\theta} \\
&= \frac{\theta}{1-\theta} \sum_{x=1}^{\infty} \left[e^t(1-\theta)\right]^x \\
&= \frac{\theta}{1-\theta} \; \frac{e^t(1-\theta)}{1-e^t(1-\theta)} \\
&= \frac{\theta e^t}{1-e^t(1-\theta)}
\end{aligned}$$

5.21 The first derivative of the MGF is

$$M'(t) = \frac{\theta e^t[1-e^t(1-\theta)] + [e^t(1-\theta)]\theta e^t}{[1-e^t(1-\theta)]^2} = \frac{\theta e^t - \theta e^{2t}(1-\theta) + \theta e^{2t} - \theta^2 e^{2t}}{[1-e^t(1-\theta)]^2} = \frac{\theta e^t}{[1-e^t(1-\theta)]^2}$$

Therefore we obtain

$$E[X] = M'(t)\Big|_{t=0} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$$

The second derivative of MGF is

$$M''(t) = \frac{\theta e^t \left[1-e^t(1-\theta)\right]^2 + 2e^t(1-\theta)\left[1-e^t(1-\theta)\right]\theta e^t}{\left[1-e^t(1-\theta)\right]^4}$$

Hence, we get

$$E[X^2] = M''(t)\Big|_{t=0} = \frac{\theta^2 - 2\theta\theta(1-\theta)}{\theta^4} = \frac{2-\theta}{\theta^2}$$

Using the above results we find the variance as follows:

$$\sigma^2 = E[X^2] - (E[X])^2 = \frac{2-\theta}{\theta^2} - \frac{1}{\theta^2} = \frac{1-\theta}{\theta^2}$$

6.6 For μ_1' we have

$$\begin{aligned}
\mu_1' &= E[X] = \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{\beta}{(x-\alpha)^2+\beta^2} dx \\
&= \frac{1}{\beta\pi} \int_{-\infty}^{\infty} \frac{x}{\left(\dfrac{x-\alpha}{\beta}\right)^2 + 1} dx
\end{aligned}$$

Now using the transformation $y = \dfrac{x-\alpha}{\beta}$ and $dx = \beta dy$ we have

$$\begin{aligned}
\mu_1' &= \frac{1}{\beta\pi} \int_{-\infty}^{\infty} \frac{\beta y + \alpha}{y^2+1} \, \beta \; dy \\
&= \frac{\beta}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2+1} \, dy + \alpha \int_{-\infty}^{\infty} \frac{1}{\pi} \, \frac{1}{y^2+1} \, dy
\end{aligned}$$

The second integral on the left side is simply the integral over the pdf of a Cauchy distribution with $\alpha = 0$ and $\beta = 1$, hence it is equal to 1. The first integral on the left side is as follows:

$$\int_{-\infty}^{\infty} \frac{y}{y^2+1}\, dy = \frac{1}{2}\ln(y^2+1)\Big]_{-\infty}^{\infty} = \infty$$

Therefore the integral does not exists. Hence μ_1' does not exist.

For μ_2' we have similar result as follows:

$$\begin{aligned}
\mu_2' &= E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\pi} \frac{\beta}{(x-\alpha)^2+\beta^2}\, dx \\
&= \frac{1}{\beta\pi}\int_{-\infty}^{\infty} \frac{x^2}{\left(\dfrac{x-\alpha}{\beta}\right)^2+1}\, dx \\
&= \frac{1}{\beta\pi}\int_{-\infty}^{\infty} \frac{(\beta y+\alpha)^2}{y^2+1}\,\beta\, dy \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta^2 y^2 + 2\beta\alpha y + \alpha^2}{y^2+1}\, dy \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\beta^2 y^2 + \beta^2}{y^2+1}\, dy + \frac{2\beta\alpha}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+1}\, dy + \frac{\beta^2+\alpha^2}{\pi}\int_{-\infty}^{\infty} \frac{1}{y^2+1}\, dy
\end{aligned}$$

As we calculated above, the second integral on the left does not exist. Hence μ_2' does not exist as well. Furthermore observe that the first integral is

$$\int_{-\infty}^{\infty} \frac{\beta^2 y^2 + \beta^2}{y^2+1}\, dy = \beta^2 \int_{-\infty}^{\infty} 1\, dy = \infty$$

6.23 (a) Since it is a pdf then the integral over all values of X must equal to 1. We have

$$1 = k\int_0^{\infty} x^{\beta-1} e^{-\alpha x^\beta} dx$$

Let

$$u = \alpha x^\beta$$

then

$$du = \alpha\beta x^{\beta-1} dx$$

hence

$$1 = k\int_0^{\infty} x^{\beta-1} e^{-\alpha x^\beta} dx = k\int_0^{\infty} \frac{1}{\alpha\beta} e^{-u} du = \frac{k}{\alpha\beta}$$

Therefore,

$$k = \alpha\beta$$

(b) Using the definition of the mean and the above transformation we have

$$\mu = E[X] = \alpha\beta\int_0^{\infty} x^\beta e^{-\alpha x^\beta} dx = \alpha^{-1/\beta}\int_0^{\infty} u^{1/\beta} e^{-u} du = \alpha^{-1/\beta}\,\Gamma\left(1+\frac{1}{\beta}\right)$$

6.26 The pdf of Beta distribution is given by

$$f(x) = kx^{\alpha-1}(1-x)^{\beta-1}$$

where $k = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$. For simplicity we use k. We then have,

$$\begin{aligned}
\frac{df}{dx} &= k(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - k(\beta-1)x^{\alpha-1}(1-x)^{\beta-2} \\
&= kx^{\alpha-2}(1-x)^{\beta-2}\left[(\alpha-1)(1-x) - (\beta-1)x\right] \\
&= kx^{\alpha-2}(1-x)^{\beta-2}\left[x(2-\alpha-\beta) + \alpha - 1\right] \\
&= 0
\end{aligned}$$

which gives either

$$[x(2-\alpha-\beta)+\alpha-1] = 0 \quad \Rightarrow \quad x = \frac{\alpha-1}{\alpha+\beta-2}$$

or $x = 0$ and $x = 1$. Checking these critical values shows that $x = \dfrac{\alpha-1}{\alpha+\beta-2}$ is actually the maximum value.

6.37 From the Normal distribution we have

$$\mathbb{E}[X] = \mu, \quad \mathbb{E}[X^2] = \sigma^2 + \mu^2, \quad \mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2.$$

Therefore we get

$$Cov[X, X^2] = E[X^3] - E[X]E[X^2] = (\mu^3 + 3\mu\sigma^2) - \mu(\sigma^2 + \mu^2) = 2\mu\sigma^2$$

In the case of standard Normal distribution $\mu = 0$, hence

$$Cov[X, X^2] = 0$$

6.39 Since we have

$$M_{X-\mu}(t) = e^{-t\mu}M_X(t)$$

then we get

$$K_X(t) = -t\mu + \ln M_X(t)$$

We also learnt that

$$M_X(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \cdots$$

therefore we get

$$\ln M_X(t) = \ln\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \cdots\right)\right]$$

Using Macluarin's series

$$\ln(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \cdots$$

we obtain the followings:

$$\begin{aligned}
K_X(t) &= -\mu t + \left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right] \\
&\quad -\frac{1}{2}\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right]^2 \\
&\quad +\frac{1}{3}\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right]^3 \\
&\quad -\frac{1}{4}\left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \cdots\right]^4 \\
&\quad + \cdots \\
&= t\left[\mu_1' - \mu\right] + \frac{t^2}{2!}\left[\mu_2' - (\mu_1')^2\right] + \frac{t^3}{3!}\left[\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3\right] \\
&\quad + \frac{t^4}{4!}\left[\mu_4' - 3(\mu_2')^2 - 4(\mu_1')(\mu_3') + 12(\mu_1')^2(\mu_2') - 6(\mu_1')^4\right] \\
&\quad + \cdots
\end{aligned}$$

(a) By looking at $K_X(t)$, we can see that $\frac{t^2}{2!}$ is multiplied by μ_2 because

$$\kappa_2 = \mu_2' - (\mu_1')^2 = \mu_2$$

(b) We also observe that

$$\kappa_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = \mu_3$$

(c) By doing some calculations we obtain

$$\begin{aligned}
\kappa_4 &= \mu_4' - 3(\mu_2')^2 - 2(\mu_1')(\mu_3') + 12(\mu_1')^2(\mu_2') - 6(\mu_1')^4 \\
&= \left[\mu_4' - 4(\mu_1')(\mu_3') + 6(\mu_1')^2(\mu_2') - 3(\mu_1')^4\right] - 3\left(\mu_2' - (\mu_1')^2\right)^2 \\
&= \mu_4 - 3\mu_2^2
\end{aligned}$$