Assignment 3 - Solutions

20th November, 2017

4.6 Since X is a discrete random variable we need to sum the probability of each value multiple by the value, that is,

$$\mathbb{E}[X] = \sum_{x\in\{-1,0,1,3\}} xf(x) = -1\frac{|-1-2|}{7} + 0\frac{|0-2|}{7} + 1\frac{|1-2|}{7} + 3\frac{|3-2|}{7} = \frac{1}{7}$$

[10 marks]

4.9 Similar to previous example, X is a discrete random variable, therefore,

(a)

$$\mathbb{E}[X] = \sum_{x\in\{0,1,2,3\}} xf(x) = 0\frac{1}{125} + 1\frac{12}{125} + 2\frac{48}{125} + 3\frac{64}{125} = \frac{12}{5} = 2.4$$

$$\mathbb{E}[X^2] = \sum_{x\in\{0,1,2,3\}} x^2 f(x) = 0^2\frac{1}{125} + 1^2\frac{12}{125} + 2^2\frac{48}{125} + 3^2\frac{64}{125} = \frac{156}{25} = 6.24$$

[5 marks]

(b) Using the expansion of $(3X+2)^2$ we have

$$\mathbb{E}\left[(3X+2)^2\right] = \mathbb{E}\left[9X^2 + 12X + 4\right] = 9\mathbb{E}[X^2] + 12\mathbb{E}[X] + 4 = 9\left(\frac{156}{25}\right) + 12\left(\frac{12}{5}\right) + 4 = 88.96$$

[5 marks]

4.25 Using the Binomial theorem to expand $(x-\mu)^r$ we get

$$\begin{aligned}
\mu_r &= \mathbb{E}[(X-\mu)^r] = \sum (x-\mu)^r f(x) \\
&= \sum x^r f(x) - \binom{r}{1}\mu \sum x^{r-1} f(x) + \cdots + (-1)^i \binom{r}{i}\mu^i \sum x^{r-i} f(x) + \cdots \\
&\quad +(-1)^{r-1}\binom{r}{r-1}\mu^{r-1}\sum x f(x) + (-1)^r \mu^r \sum f(x) \\
&= \mu'_r - \binom{r}{1}\mu\mu'_{r-1} + \cdots + (-1)^i\binom{r}{i}\mu^i\mu'_{r-i} + \cdots + (-1)^{r-1}(r-1)\mu^r
\end{aligned}$$

where

$$\mu'_r = \sum_x x^r f(x)$$

(a) Using the above equality for $\mu_3 = \mathbb{E}[(X-\mu)^3]$ with $r=3$ we get

$$\mu_3 = \mu_3^{'} - \binom{3}{1}\mu\mu_2^{'} + \binom{3}{2}\mu^2\mu_1^{'} - \mu^3 = \mu_3^{'} - 3\mu\mu_2^{'} + 3\mu^2\mu - \mu^3 = \mu_3^{'} - 3\mu\mu_2^{'} + 2\mu^3$$

[5 marks]

(b) Using the above equality for $\mu_4 = \mathbb{E}[(X-\mu)^4]$ with $r=4$ we get

$$\begin{aligned} \mu_4 &= \mu_4^{'} - \binom{4}{1}\mu\mu_3^{'} + \binom{4}{2}\mu^2\mu_2^{'} - \binom{4}{3}\mu^3\mu_1^{'} + \mu^4 \\ &= \mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 4\mu^3\mu + \mu^4 \\ &= \mu_4^{'} - 4\mu\mu_3^{'} + 6\mu^2\mu_2^{'} - 3\mu^4 \end{aligned}$$

[5 marks]

4.32 Chebyshev's theorem states that

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Let $k\sigma = c$, then

$$P(|x-\mu| < c) \geq 1 - \frac{\sigma^2}{c^2}$$

[10 marks]

4.38 Using the Maclaurin series of the function $\frac{1}{1-t^2}$, we get

$$M_x(t) = 1 + t^2 + t^4 + \cdots$$

(a) Since $\sigma_X^2 = \mathbb{E}[X^2] - (\mu_X)^2$ and by comparing the above equation with the general moment generating function

$$M_X(t) = 1 + \mu t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots$$

we get

$$\mu_X = \mu = 0,$$

and

$$\mathbb{E}[X^2] = \mu_2' = \frac{1}{\frac{1}{2!}} = 2! = 2,$$

then,

$$\sigma_X^2 = \mathbb{E}[X^2] - (\mu_X)^2 = 2$$

[5 marks]

(b) Since $\sigma_X^2 = \mathbb{E}[X^2] - (\mu_X)^2$, and

$$M_X'(t) = (-1)(1-t^2)^{-2}(-2t) = \frac{2t}{(1-t^2)^2},$$

$$M_X''(t) = \frac{2(1-t^2)^2 - 2t(2)(1-t^2)(-2t)}{(1-t^2)^4},$$

then

$$\sigma_X^2 = M_X''(t)\Big|_{t=0} - \left[M_X'(t)\Big|_{t=0}\right]^2 = 2 - 0 = 2$$

[5 marks]

4.44 We know that

$$Cov(X_1, X_3) = \mathbb{E}[X_1X_3] - \mathbb{E}[X_1]\mathbb{E}[X_3]$$

and since

$$f(x_1, x_3) = \left(x_1 + \frac{1}{2}\right) e^{-x_3}, \quad f(x_1) = x_1 + \frac{1}{2}, \quad f(x_3) = e^{-x_3}$$

we have

$$\begin{aligned}
\mathbb{E}[X_1X_3] &= \int_0^1 \int_0^\infty x_1x_3 \left(x_1 + \frac{1}{2}\right) e^{-x_3} dx_3dx_1 = \int_0^1 x_1 \left(x_1 + \frac{1}{2}\right) \int_0^\infty x_3e^{-x_3} dx_3dx_1 \\
&= \int_0^1 x_1 \left(x_1 + \frac{1}{2}\right) dx_1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}
\end{aligned}$$

Using the marginal distribution of X_1 we get

$$\mathbb{E}[X_1] = \int_0^1 x_1 \left(x_1 + \frac{1}{2}\right) dx_1 = \frac{7}{12}$$

Also using the marginal distribution of X_3 we get

$$\mathbb{E}[X_3] = \int_0^\infty x_3e^{-x_3} dx_3 = 1$$

and finally we have

$$Cov(X_1, X_3) = \frac{7}{12} - \frac{7}{12}(1) = 0$$

[10 marks]

4.48 (a)

$$\mathbb{E}\left[e^{t_1X_1+t_2X_2+\cdots+t_kX_k}\right] = \int \cdots \int e^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k$$

Then the derivative with respect to t_i is

$$\frac{\partial \int \cdots \int e^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k}{\partial t_i} = \int \cdots \int x_ie^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k$$

If $t_1 = t_2 = \cdots = t_k = 0$ then

$$\begin{aligned}
\left.\frac{\partial \int \cdots \int e^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k}{\partial t_i}\right|_{t_1=\cdots=t_k=0} &= \int \cdots \int x_if(x_1, \ldots, x_k)dx_1 \cdots dx_k \\
&= \mathbb{E}[X_i]
\end{aligned}$$

[4 marks]

(b)

$$\frac{\partial^2 \int \cdots \int e^{\sum_{i=1}^k t_ix_i} f(x_1, ..., x_k)dx_1...dx_k}{\partial t_i\partial t_j} = \int \cdots \int x_ix_je^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k$$

If $t_1 = t_2 = \cdots = t_k = 0$ then

$$\begin{aligned}
\left.\frac{\partial^2 \int \cdots \int e^{\sum_{i=1}^k t_ix_i} f(x_1, \ldots, x_k)dx_1 \cdots dx_k}{\partial t_i\partial t_j}\right|_{t_1=\cdots=t_k=0} &= \int \cdots \int x_ix_jf(x_1, \ldots, x_k)dx_1 \cdots dx_k \\
&= \mathbb{E}[X_iX_j]
\end{aligned}$$

[3 marks]

(c)

$$\begin{aligned} M_{XY}(t_1,t_2) &= \mathbb{E}[e^{t_1x+t_2y}] = \int_0^\infty \int_0^\infty e^{t_1x+t_2y} e^{-x-y} dxdy \\ &= \int_0^\infty \int_0^\infty e^{x(t_1-1)} e^{y(t_2-1)} dxdy = \frac{1}{t_1-1} \frac{1}{t_2-1} \end{aligned}$$

$$\frac{\partial M}{\partial t_1} = \frac{-1}{(t_1-1)^2} \frac{1}{t_2-1}$$

If $t_1 = 0 \Rightarrow \mathbb{E}[X] = 1$.
Since $M_{XY}(t_1,t_2)$ is symmetrical setting $t_2 = 0$ gives $\mathbb{E}[Y] = 1$.

$$\frac{\partial^2 M}{\partial t_1 \partial t_2} = \frac{1}{(t_1-1)^2(t_2-1)^2}$$

If $t_1 = t_2 = 0 \Rightarrow \mathbb{E}[XY] = 1$

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 1 - 1 = 0$$

[3 marks]

4.51

$$Var(W) = 9Var(X) + 16Var(Y) + 2Cov(3X,4Y) = 9Var(X) + 16Var(Y) + 24Cov(X,Y)$$

$$\mathbb{E}[X] = \frac{1}{3}\int_0^1 \int_0^2 (x^2+xy)dydx = \frac{1}{3}\int_0^1 (2x^2+2x)dx = \frac{1}{3}(\frac{2}{3}+1) = \frac{5}{9}$$

$$\mathbb{E}[X^2] = \frac{1}{3}\int_0^1 \int_0^2 (x^3+x^2y)dydx = \frac{1}{3}\int_0^1 (2x^3+2x^2)dx = \frac{1}{3}(\frac{1}{2}+\frac{2}{3}) = \frac{7}{18}$$

$$Var(X) = \frac{7}{18} - \frac{25}{81} = \frac{13}{162}$$

$$\mathbb{E}[Y] = \frac{1}{3}\int_0^2 \int_0^1 (xy+y^2)dxdy = \frac{1}{3}\int_0^2 (\frac{1}{2}y+y^2)dy = \frac{1}{3}(1+\frac{8}{3}) = \frac{11}{9}$$

$$\mathbb{E}[Y^2] = \frac{1}{3}\int_0^2 \int_0^1 (xy^2+y^3)dxdy = \frac{1}{3}\int_0^2 (\frac{1}{2}y^2+y^3)dy = \frac{1}{3}(\frac{4}{3}+4) = \frac{16}{9}$$

$$Var(Y) = \frac{16}{9} - \frac{121}{81} = \frac{23}{81}$$

$$\mathbb{E}[XY] = \frac{1}{3}\int_0^1 \int_0^2 (yx^2+xy^2)dydx = \frac{1}{3}\int_0^1 (2x^2+\frac{8}{3}x)dx = \frac{1}{3}(\frac{2}{3}+\frac{4}{3}) = \frac{2}{3}$$

$$Cov(X,Y) = \frac{2}{3} - \frac{5}{9}\frac{11}{9} = \frac{-1}{81}$$

$$Var(W) = 9\frac{13}{162} + 16\frac{23}{81} + 24\frac{-1}{81} = \frac{805}{162}$$

[10 marks]

4.58

$$f(x) = \frac{1}{4}\int_0^2 (2x+y)dy = \frac{1}{4}(4x+2) = \frac{1}{2}(2x+1)$$

$$f\left(y \mid x = \frac{1}{4}\right) = \frac{f_{xy}\left(y\frac{1}{4}\right)}{f_x\left(\frac{1}{4}\right)} = \frac{\frac{1}{4}\left(\frac{1}{2}+y\right)}{\frac{1}{2}\frac{3}{2}} = \frac{1}{6}(2y+1)$$

$$Var\left[Y \mid x = \frac{1}{4}\right] = \mathbb{E}\left[Y^2 | x = \frac{1}{4}\right] - \mathbb{E}\left[Y | x = \frac{1}{4}\right]^2$$

$$\mathbb{E}\left[Y \mid x = \frac{1}{4}\right] = \frac{1}{6}\int_0^2 \left(2y^2+y\right)dy = \frac{1}{6}\left(\frac{16}{3}+2\right)\frac{1}{6}\frac{22}{3} = \frac{11}{9}$$

$$\mathbb{E}\left[Y^2 \mid x = \frac{1}{4}\right] = \frac{1}{6}\int_0^2 (2y^3+y^2)dy = \frac{1}{6}\left(8+\frac{8}{3}\right) = \frac{1}{6}\left(\frac{32}{3}\right) = \frac{16}{9}$$

$$Var\left[Y \mid x = \frac{1}{4}\right] = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

[10 marks]

4.60 (a) Using the definition of cumulative function we have

$$F(x \mid a \le x \le b) \quad = \quad P(X \le x \mid a \le X \le b)$$

Using the definition of conditional probability we have

$$P(X \le x \mid a \le X \le b) \quad = \quad \frac{P(X \le x, a \le X \le b)}{P(a \le X \le b)}$$

Defining $A = \{X \mid X \le x\}$ and $B = \{X \mid a < X \le b\}$ then the nominator is intersection of two events and is as follows:

$$\begin{array}{lll} A \cap B = \phi & \text{if} & x < a \\ A \cap B = \{a < X \le x\} & \text{if} & a < x \le b \\ A \cap B = B & \text{if} & b \le x \end{array}$$

Therefore,

$$P(X \le x \mid a < X \le b) = \frac{P(\phi)}{P(a < X \le b)} = 0 \qquad \text{if} \qquad x < a$$

$$P(X \le x \mid a < X \le b) = \frac{P(a < X \le x)}{P(a < X \le b)} = \frac{F(x)-F(a)}{F(b)-F(a)} \qquad \text{if} \qquad a < x \le b$$

$$P(X \le x \mid a < X \le b) = \frac{P(a < X \le b)}{P(a < X \le b)} = 1 \qquad \text{if} \qquad b < x$$

Hence,

$$F(x \mid a \le x \le b) = \begin{cases} 0 & \text{if} \quad x < a \\ \frac{F(x)-F(a)}{F(b)-F(a)} & \text{if} \quad a < x \le b \\ 1 & \text{if} \quad b < x \end{cases}$$

[5 marks]

(b) Using the above cumulative distribution, we can get the conditional density as follows:

$$f(x \mid a \leq x \leq b) = \begin{cases} \dfrac{f(x)}{F(b) - F(a)} & \text{if} \quad a < x \leq b \\ 0 & \text{if} \quad \text{elsewhere} \end{cases}$$

Therefore,

$$\mathbb{E}[u(X)|a \leq x \leq b] = \int_a^b \frac{u(x)f(x)}{F(b) - F(a)} dx = \frac{\int_a^b u(x)f(x)dx}{F(b) - F(a)} = \frac{\int_a^b u(x)f(x)dx}{\int_a^b f(x)dx}$$

[5 marks]