Assignment 2

① 3.4 pg. 72 8 marks

a) $c(1+2+3+4+5)=1 \Rightarrow c=\frac{1}{15}$

b) $c\left[\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\dots+\binom{5}{5}\right]=c\cdot\left[1+5+10+10+5+1\right]=32c=1 \Rightarrow c=\frac{1}{32}$

c) $\sum_{x=1}^{K} cx^2 = c\cdot\sum_{x=1}^{K} x^2 = c\cdot\frac{K(K+1)(2K+1)}{6}=1 \Rightarrow c=\frac{6}{K(K+1)(2K+1)}$

d) $\sum_{x=1}^{\infty} c\left(\frac{1}{4}\right)^x = c\,\frac{\left(1-\lim_{n\to\infty}\left(\frac{1}{4}\right)^n\right)\frac{1}{4}}{1-\frac{1}{4}} = c\cdot\frac{\frac{1}{4}}{\frac{3}{4}} = c\cdot\frac{1}{3}=1 \Rightarrow c=3$

② 3.13 p. 73 12 marks

a) $P(X\le 3)=\frac{3}{4}$

b) $P(X=3)=P(X\le 3)-P(X<3)=F(3)-P(X\le 1)=F(3)-F(1)=\frac{3}{4}-\frac{1}{2}=\frac{1}{4}$

c) $P(X<3)=P(X\le 1)=F(1)=\frac{1}{2}$

d) $P(X\ge 1)=1-P(X<1)=1-P(X\le -1)=1-\frac{1}{4}=\frac{3}{4}$

e) $P(-0.4<X<4)=P(X<4)-P(X\le -0.4)=P(X\le 3)-P(X\le -1)=F(3)-F(-1)=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$

f) $P(X=5)=P(X\le 5)-P(X<5)=F(5)-P(X\le 3)=F(5)-F(3)=1-\frac{3}{4}=\frac{1}{4}$

③ 3.15 pg. 73 6 marks Given that $P(A) = 1 - P(A^c)$

a) $P(X > x_i) = 1 - P(X \leq x_i) = 1 - F(x_i) \quad i = 1, 2, \ldots, n$

b) $P(X \geq x_i) = 1 - P(X < x_i) = 1 - P(X \leq x_{i-1}) = 1 - F(x_{i-1}) \quad i = 2, \ldots, n$ and $P(X \geq x_1) = 1$
$(P(X \geq x_1) = 1 - F(-\infty) = 1 - 0 = 1)$

④ 3.22 pg. 80 6 marks

a) $1 = \int_0^4 \frac{c}{\sqrt{x}} dx = c \left. \frac{x^{\frac{1}{2}}}{1/2} \right|_0^4 = 2c \cdot 2 = 4c \Rightarrow c = \frac{1}{4}$

b) $P(X < \frac{1}{4}) = \frac{1}{4} \int_0^{1/4} \frac{1}{\sqrt{x}} dx = \frac{1}{4} \cdot 2 \sqrt{x} \Big|_0^{\frac{1}{4}} = \frac{1}{4}$

$P(X > 1) = 1 - \int_0^1 \frac{1}{4} \frac{1}{\sqrt{x}} dx = 1 - \frac{1}{2} \sqrt{x} \Big|_0^1 = \frac{1}{2}$

⑤ 3.32 pg. 81 4 marks Because $F(x)$ is a continuous function $P(X < x_i) = P(X \leq x_i$

$P(-\frac{1}{2} < X < \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

$P(2 < X < 3) = F(3) - F(2) = 1 - 1 = 0$

⑥ 3.37 pg. 81 4 marks

$P(x \leq 2) = F(2) = 1 - 3e^{-2} = 0.59$

$P(1 < X < 3) = F(3) - F(1) = 1 - 4e^{-3} - (1 - 2e^{-1}) = 0.536$

$P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1 - (1+4)e^{-4}) = 5e^{-4} = 0.0915$

⑦ 3.42 pg. 90 8 marks

a) $P(X=1, Y=2) = \frac{1}{20}$

b) $P(X=0, 1 \leq Y < 3) = P(X=0, Y=1) + P(X=0, Y=2)$
$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

c) $P(X+Y \leq 1) = P(X=0, Y=0)$
$+ P(X=0, Y=1) + P(X=1, Y=0)$
$= \frac{1}{12} + \frac{1}{6} + \frac{1}{4} = \frac{1}{2}$

d) $P(X > Y) = P(X=1, Y=0) + P(X=2, Y=0) + P(X=2, Y=1)$
$= \frac{1}{6} + \frac{1}{24} + \frac{1}{40} = \frac{7}{30}$

(8) 3.44 pg. 90 2 marks

$$c(2+5+10+1+4+9+2+5+10+10+13+18)=1 \Rightarrow c=\frac{1}{89}$$

(9) 3.48 pg. 90 6 marks

a) $F(-\infty,-\infty)=P(X\le-\infty, Y\le -\infty)=0$

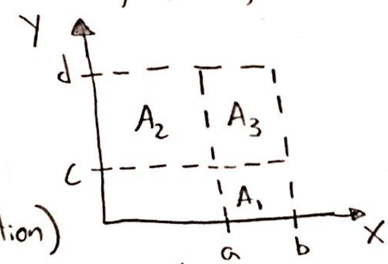
b) $F(\infty,\infty)=P(X\le\infty,Y\le\infty)=1$

$a<b \quad c<d$

c) $F(b,d)=P(X\le b, Y\le d) = F(a,c) + \overbrace{P(a<X\le b, Y\le c)}^{A_1} + \overbrace{P(X\le a, c<Y\le d)}^{A_2}$

$+\overbrace{P(a<X\le b, c<Y\le d)}^{A_3}$

$\Rightarrow F(b,b)\ge F(a,c)$ ■



(10) 3.50 pg. 90 4 marks (Continuous function)

$$P(X+Y<\tfrac12) = P(Y<\tfrac12 - X) = \int_0^{1/2}\int_0^{1/2-x} 24xy\,dy\,dx = 24\int_0^{1/2} \left. x\frac{y^2}{2}\right|_0^{\frac12 -x} dx = 12\int_0^{1/2} x\left(\tfrac12-x\right)^2dx$$

$$=12\int_0^{1/2}\left(\frac{x}{4}-x^2+x^3\right)dx = 12\left[\frac{x^2}{8}-\frac{x^3}{3}+\frac{x^4}{4}\right]_0^{1/2} = 12\left[\frac1{32}-\frac1{24}+\frac1{64}\right] = \frac1{16}$$

(11) 3.68 pg. 91 10 marks

a) $P(X=\frac12, Y=\frac12, Z=\frac12)=0$

b) $P(X<\frac12,Y<\frac12,Z<\frac12) = \frac13\int_0^{\frac12}\int_0^{\frac12}\int_0^{\frac12}(2x+3y+z)\,dz\,dy\,dx = \frac13\int_0^{\frac12}\int_0^{\frac12} \left. 2xz+3yz+\frac{z^2}{2}\right|_0^{1/2} dy\,dz$

$$=\frac13\int_0^{1/2}\int_0^{1/2}\left(x+\frac32 y+\frac18\right)dy\,dx = \frac13\int_0^{1/2} \left. xy+\frac34y^2+\frac18 y\right|_0^{1/2}dx = \frac13\int_0^{1/2}\frac{x}{2}+\frac3{16}+\frac1{16}\,dx = \frac13\left[\frac{x^2}{4}+\frac4{16}x\right]_0^{1/2}$$

$$=\frac13\left[\frac1{16}+\frac2{16}\right]=\frac1{16}$$

(12) 3.71 pg. 100 10 marks

$$f(x,y,z) = \frac{xyz}{108}$$

a) $$f(x,y) = \frac{xy}{108}(1+2) = \frac{xy}{36}$$

b) $$f(x,z) = \frac{xz}{108}(1+2+3) = \frac{xz}{18}$$

c) $$f(x) = \frac{x}{12}(1+2) = \frac{x}{6}$$

d) $$f(z \mid x=1, y=2) = \frac{z/54}{2/36} = \frac{z}{3}$$

e) $$f(y,z \mid x=3) = \frac{yz/36}{1/2} = \frac{yz}{18}$$

(13) 3.76 pg. 100 10 marks

a) $$f(x) = \int_0^{1-x} 24y(1-x-y)\,dy = 24\int_0^{1-x}(y - xy - y^2)\,dy = 24\left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3}\right]\Big|_0^{1-x} = 12(1-x)^2 - 12x(1-x)^2 - 8(1-x)^3$$

$$= 12(1-x)^2(1-x) - 8(1-x)^3 = 4(1-x)^3 \quad 0 \le x \le 1$$

b) $$f(y) = \int_0^{1-y} 24y(1-x-y)\,dx = 24\left[xy - y\frac{x^2}{2} - y^2x\right]\Big|_0^{1-y} = 24y(1-y) - 12y(1-y)^2 - 24y^2(1-y)$$

$$= 24y(1-y)(1-y) - 12y(1-y)^2 = 12y(1-y)^2$$

$f(x,y) \neq f(x)\cdot f(y) \Rightarrow$ Not independent.

(14) 3.80 pg. 101 10 marks

$$M(x_1,x_3) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_3}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_2\,dx_3\,dx_1 = F(x_1,\infty,x_3)$$

$$G(x_1) = \int_{-\infty}^{x_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)\,dx_2\,dx_3\,dx_1 = F(x_1,\infty,\infty)$$

$$f(x_1,x_2,x_3) = \begin{cases} (x_1+x_2)e^{-x_3} & 0<x_1<1,\ 0<x_2<1,\ x_3>0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x_1,x_3) = \int_0^1 (x_1+x_2)e^{-x_3}\,dx_2 = x_1e^{-x_3} + e^{-x_3}\int_0^1 x_2\,dx_2 = x_1e^{-x_3} + \frac{e^{-x_3}}{2} = e^{-x_3}\left(x_1+\frac{1}{2}\right)$$

$$f(x_1) = \int_0^\infty\int_0^1 (x_1+x_2)e^{-x_3}\,dx_2\,dx_3 = \int_0^\infty \frac{1}{2}e^{-x_3}(2x_1+1)\,dx_3 = \frac{1}{2}(2x_1+1)(-e^{-x_3})\Big|_0^\infty$$

$$= \frac{1}{2}(2x_1+1) = x_1+\frac{1}{2}$$

$$F(x_1,x_3) = \int_0^{x_1}\int_0^{x_3} e^{-x_3}\left(x_1+\frac{1}{2}\right)dx_3\,dx_1 = \int_0^{x_1}\left(x_1+\frac{1}{2}\right)\left[-e^{-x_3}\right]\Big|_0^{x_3}dx_1$$

$$= \int_0^{x_1}(1-e^{-x_3})\left(x_1+\frac{1}{2}\right)dx_1 = (1-e^{-x_3})\left[\frac{x_1^2}{2}+\frac{x_1}{2}\right]\Big|_0^{x_1} = (1-e^{-x_3})\left(\frac{x_1^2}{2}+\frac{x_1}{2}\right)$$

$$= M(x_1,x_3)$$

$$F(x_1) = G(x_1) = \int_0^{x_1}\left(x_1+\frac{1}{2}\right)dx_1 = \frac{x_1^2}{2}+\frac{x_1}{2} = \frac{x_1}{2}(x_1+1)$$