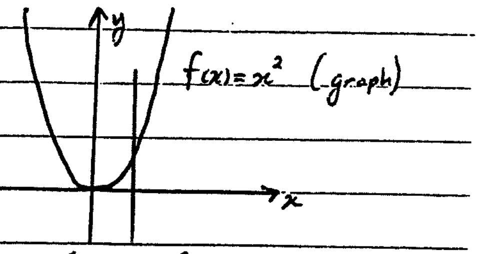
Chapter 1: Functions

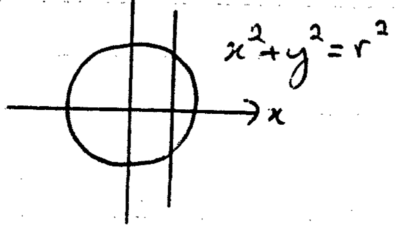
§1.1: Functions and their Graphs

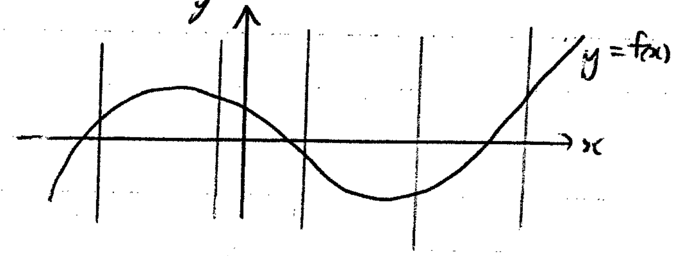


we'll begin with some definitions:
consider the function $f(x) = x^2$ (p1)
what is a function? a function is a rule that takes certain input numbers (called the independent variable) and produces unique output numbers for each input (p1)
(the values of) the output is called the dependent variable (because it depends on the independent variable value)
so, for $f(x)=x^2$, x is the independent variable, $f(x)$ is the dependent
and $f(x)=x^2$ produces a unique output for each input (this means that $f(x)$ is unique for any x, it does not mean the $f(x)$ values are all distinct)
the set of all possible inputs is called the domain of the function and the set of all outputs is called the range (p1)
so, for $f(x)=x^2$, the domain is all real x ($x \in \mathbb{R}$, $-\infty < x < \infty$)
and the range is nonnegative real numbers $f(x) \geq 0$ ($[0,\infty)$)

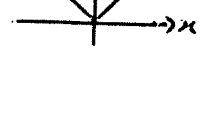
the graph of a function is the set of points $\{(x, f(x)) \mid x \in D\}$ (p3)

a curve in the xy plane is a function if no vertical line intersects the curve more than once (p 4)





sometimes, functions are defined piecewise (p5)
eg. $|x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$



(2)

some functions have symmetry properties (p6)

$f(x)$ is called *even* if $f(-x) = f(x)$ for all x

examples: $x^2, x^4, x^{2n}, \cos(x), x^2+2$, etc...

even functions are symmetric about the y axis

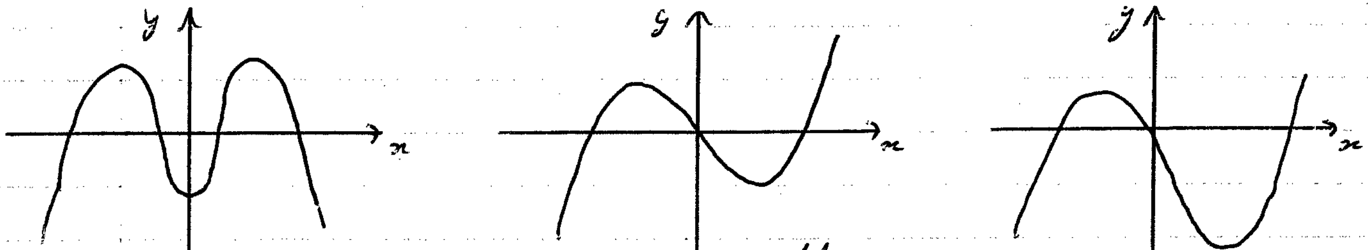
$f(x)$ is called *odd* if $f(-x) = -f(x)$ for all x

examples: $x, x^3, x^{2k+1}, \sin x, 2x^3+7x$, etc...

odd functions are symmetric about the origin

note: most functions are neither even nor odd

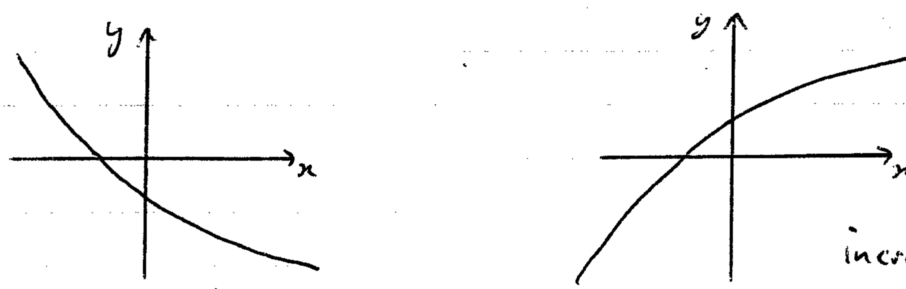
example: $f(x) = x^2 + x - 1$ contains both even and odd powers



even odd (neither)

a function is called *increasing* on an interval I if $f(x_1) < f(x_2)$ for $x_1 < x_2$ in I

and it's *decreasing* if $f(x_2) < f(x_1)$ (p6)



increasing

decreasing

one of the simplest types of functions is a *linear function* (p7)

linear functions are characterized by a constant rate of increase/decrease

ie they have constant slope (or rate of change)

they can be represented by the formula $y = mx + b$

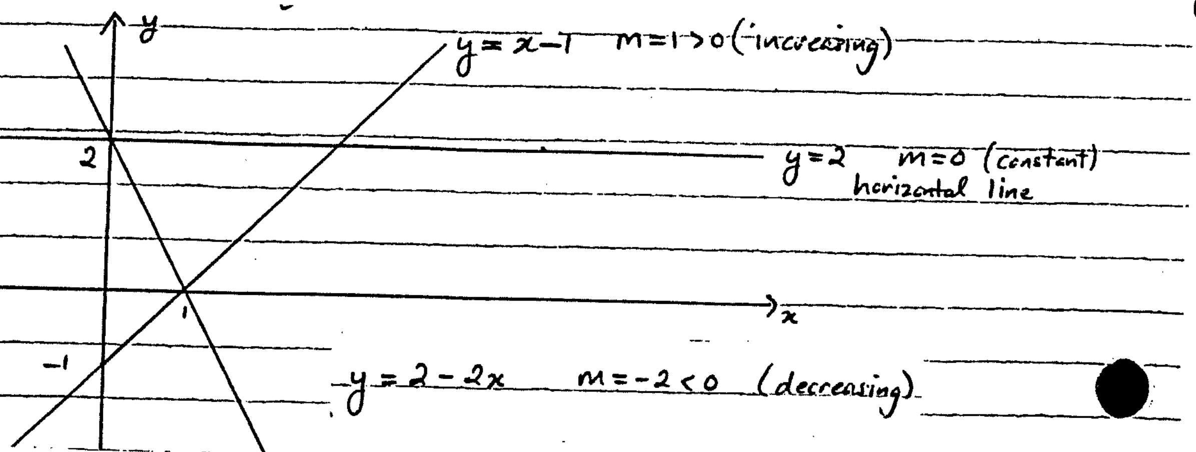
where $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$ (recall the Δ notation for change)

(difference quotient)

is the *slope* of the line

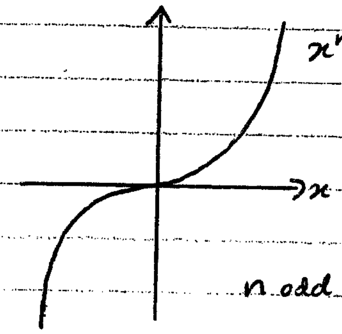
and b is the *y-intercept*

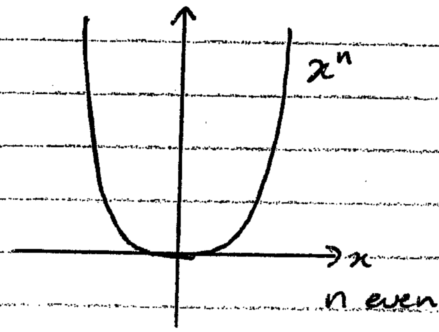
examples :



a power function has the form $f(x) = kx^p$ (p 7-8)
for constants k and p

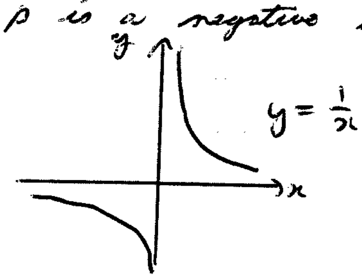
for positive integer powers, say $f(x) = x^n$, there are 2 basic shapes, depending on whether n is even or odd

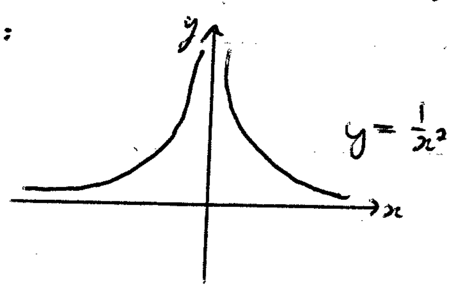




the larger n is the steeper the curves are as $x \to \pm\infty$ and the flatter near $x = 0$

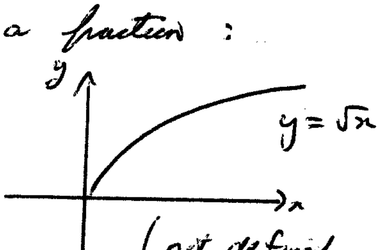
if p is not a positive integer, there are other shapes :
if p is a negative integer :



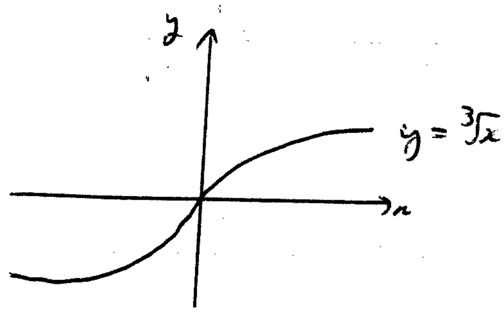


(not defined at $x = 0$)

if p is a fraction :



(not defined for $x < 0$)

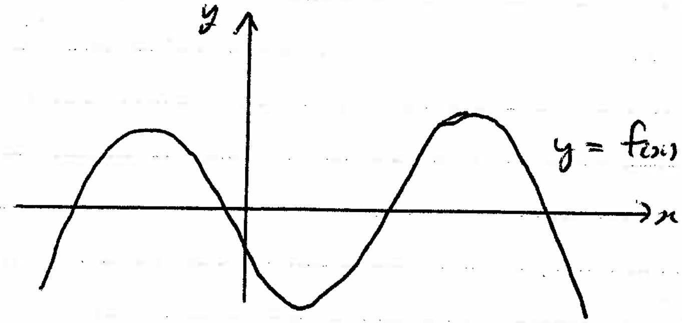


a *polynomial of degree n* has the form (p 8-9)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$ is called the *leading coefficient*

a polynomial of degree n has at most n roots or zeros and $n-1$ "turning points" (see figures on p 9)



what degree ?

a *rational function* has the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials (p 9)

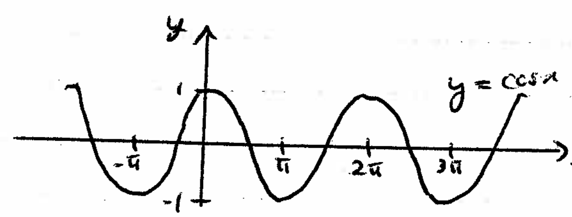
example : $f(x) = \dfrac{2x^2+3}{x^2+7x-1}$

an *algebraic* function (p 10) involves algebraic operations (eg roots)

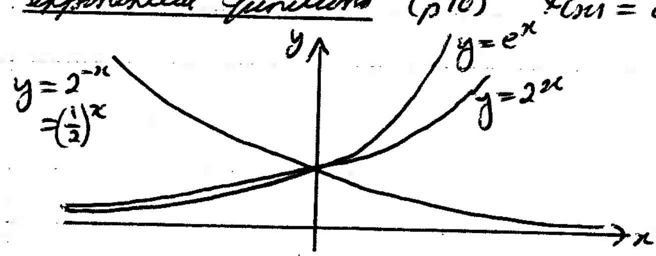
example : $f(x) = \sqrt{\dfrac{2x^2+3}{x^2+7x-1}}$

transcendental functions (p 11) are those that are not algebraic. They include :

trigonometric functions (p 10)

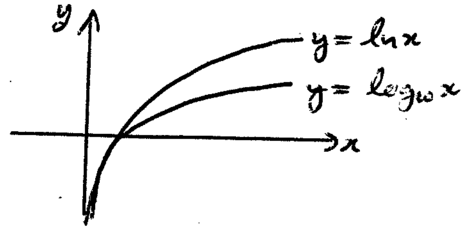


exponential functions (p 10) $f(x) = a^x$ (base $a > 0$) $(a \neq 1)$



recall $a^x \neq 0$
in fact $a^x > 0$ for all x

<u>logarithm functions</u> (p 11) (the inverses of exponentials)



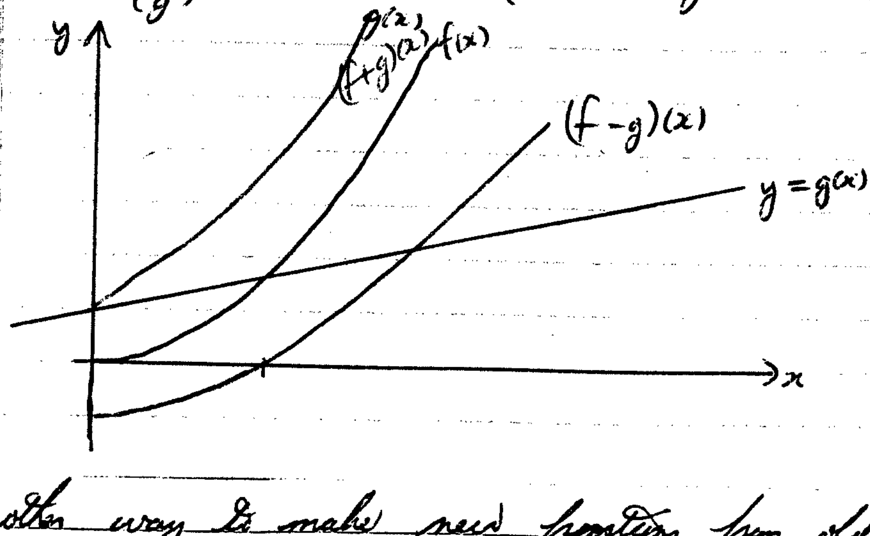
see the examples and figures throughout the section

<u>§ 1.2 : Combining Functions ; Shifting and Scaling Graphs</u>

given two functions, $f(x)$ and $g(x)$,

we can ~~also~~ add, subtract, multiply and divide

$$(f+g)(x) = f(x) + g(x) \qquad \text{(p 14-15)}$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)\,g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{domain} : g(x) \neq 0)$$



<u>another way to make new functions from old is to compose them
ie use the output of one function as the input of another (p 15)</u>

$$(f \circ g)(x) = f(g(x))$$

outer inner

the domain is all x in domain of g such that $g(x)$ is in domain of f

example: blowing up a spherical balloon

the volume is $V = \frac{4}{3}\pi r^3 = f(r)$

but the radius is a function of time, say $r = g(t) = 2t^2$

so V is really a function of t

ie $V(t) = \underbrace{f}_{\text{outer function}}(\underbrace{g(t)}_{\text{inner function}}) = \frac{4}{3}\pi (2t^2)^3 = \frac{4}{3}\pi\, 8t^6 = \frac{32}{3}\pi t^6$

example: let $f(x) = x^2+2$, $g(x) = \cos(2x)$

then $h(x) = f(g(x)) = f(\cos(2x)) = (\cos(2x))^2 + 2 = \cos^2(2x) + 2$

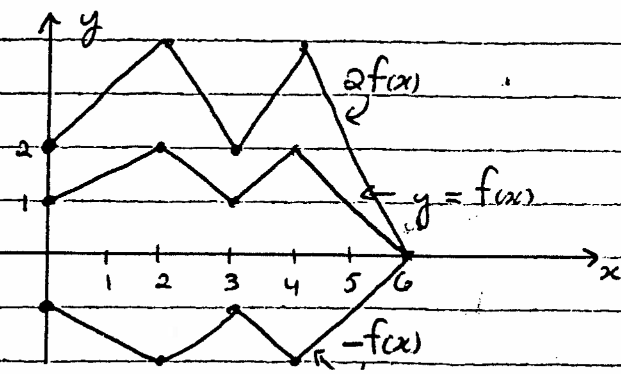
but $p(x) = g(f(x)) = \cos(2(x^2+2)) = \cos(2x^2+4)$

ie $f(g(x)) \neq g(f(x))$ (in general)

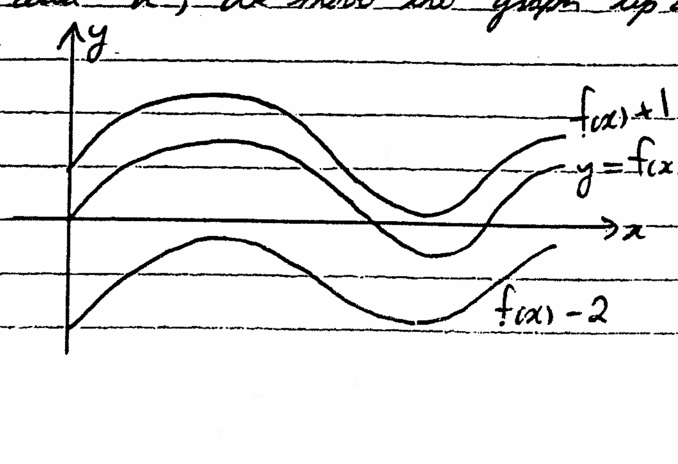
see Ex 2 p16

we can also shift and scale functions: (p 16 - 17)

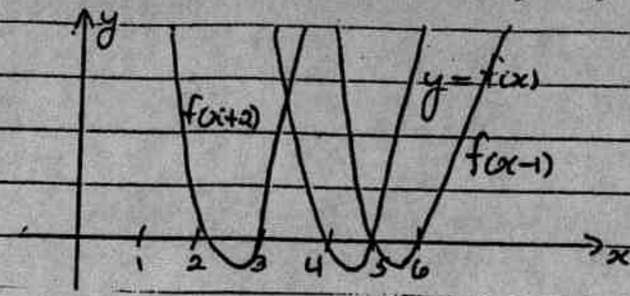
- if we multiply by a constant c, $cf(x)$, the graph will stretch if $|c|>1$, shrink if $|c|<1$ and be flipped upside down if $c<0$



- if we add k, we move the graph up or down



- if we replace x by $x-h$, the graph moves to the right by h if $h>0$ or to the left by h if $h<0$



see Ex 3-5 p 16-18

§1.3 : Trigonometric Functions

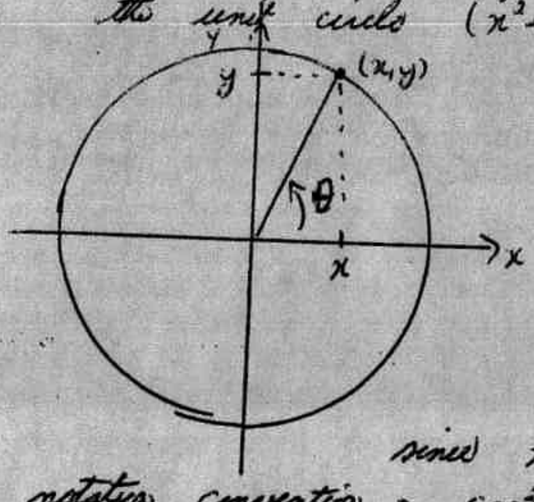
in calculus, we measure angles in radians, not degrees (there are good reasons for this that we'll see later)

an angle of 1 radian is defined to be the angle at the centre of the unit circle that cuts off an arclength of 1

since the circumference of the unit circle is 2π, the total angle contained is 2π radians (p 21)

ie $360° = 2\pi$ radians or $180° = \pi$ radians

we can define the trigonometric functions with reference to the unit circle ($x^2+y^2=1$)



at an angle of θ radians (measured counterclockwise from the x axis), we have

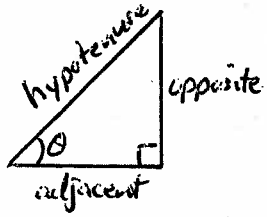
$\cos\theta = x$ (p 22)

$\sin\theta = y$

since $x^2+y^2=1$, $\cos^2\theta + \sin^2\theta = 1$

notation convention : $\cos^2\theta = (\cos\theta)^2$, $\sin^2\theta = (\sin\theta)^2$

or we can define the functions with a right triangle (p 22)



$\cos\theta = \frac{adj}{hyp}$, $\sin\theta = \frac{opp}{hyp}$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$

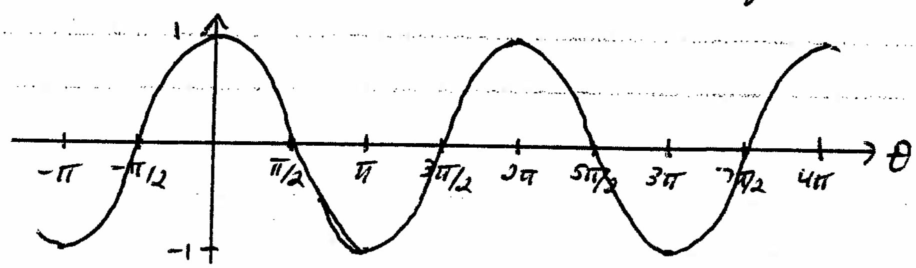
and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$

at the special angles, we get the following values :
(p23) ...

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1

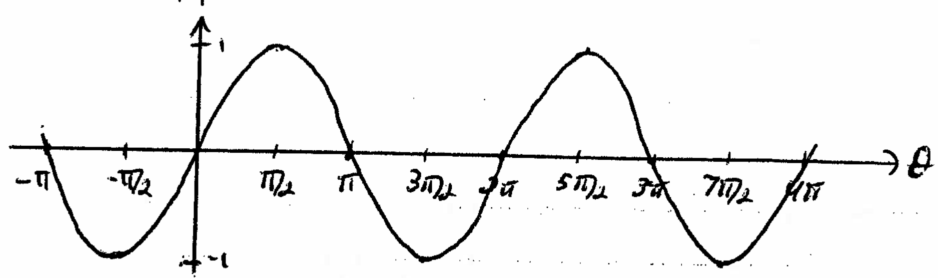
and so on ...

if we follow the values as we go around the circle (which we can do several times), we'll get the following graphs



$\cos\theta$

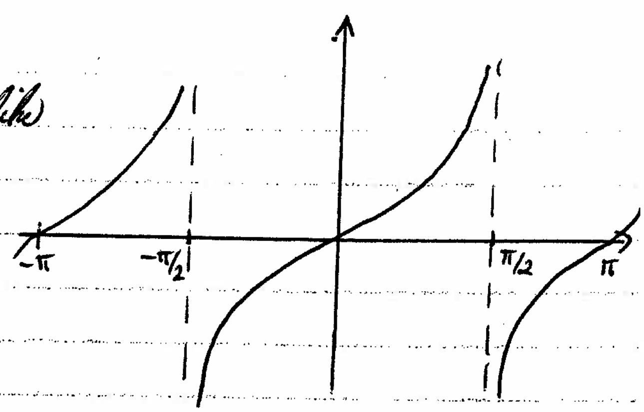
(2π periodic)



$\sin\theta$

the graph of $\tan\theta$ looks like

($\tan\theta$ is π periodic)



a function $f(x)$ is called <u>periodic with period p</u> if p is the smallest number such that
$f(x+p) = f(x)$ for all x (p 24)

recall some basic trig identities (p 24-25)

$\cos^2\theta + \sin^2\theta = 1$ (from unit circle)

divide by $\cos^2\theta$ to get $1 + \tan^2\theta = \sec^2\theta$

$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

so, in particular, if $\theta = \varphi$ $\quad \cos(2\theta) = \cos^2\theta - \sin^2\theta$

and $\quad \sin(2\theta) = 2\sin\theta\cos\theta$

substitute to find that $\quad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$

and $\quad \sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

we can apply all of the shifts and scaling to trig functions

$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$ $\quad |A|$ amplitude $\quad C$ horiz. shift

$|B|$ period $\quad D$ vert. shift

(see p 26)

§1.5: Exponential Functions

another important type of function is an exponential function (p 34)

exponential functions are characterized by growth (or decay) by a constant factor for equal intervals of the independent variable

example: $y = f(x) = 2^x$ doubles everytime x increases by 1

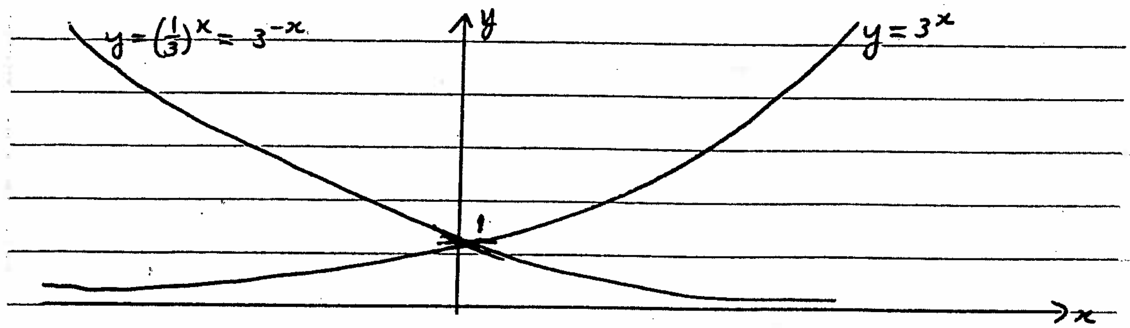
x	2^x
0	1
1	2
2	4
3	8 etc...

exponential functions can be represented by the formula $f(x) = a^x$

where a is any positive real number (and is the growth/decay factor)

if $a > 1$, the function grows (increases)

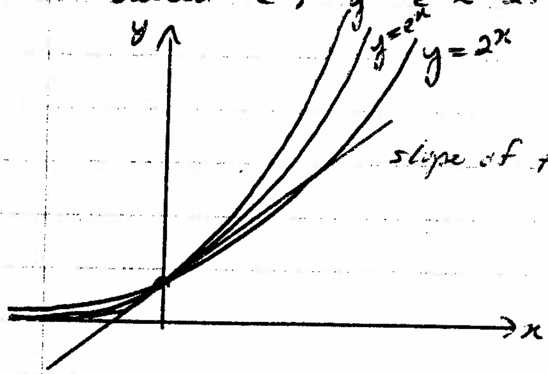
if $0 < a < 1$, the function decays (decreases)



notice that whether we have an increasing or decreasing exponential, the curve bends upwards (as we move to the right)
a graph that bends upwards is called concave up
one that bends downwards is called concave down
(a line does not bend and so is neither)
note that concave up/down is independent of increasing/decreasing

basic laws of exponents : (p36)
$a^{x+y} = a^x a^y$, $a^{x-y} = a^x/a^y$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$
$(a/b)^x = a^x/b^x$

recall that there is the special base of exponentials, the number called e, $e \approx 2.7182818\ldots$ (p36)



examples:

i, the population of a city is 600,000 and growing at a rate of 2.1 % per year
what is the function that describes the population and what will the population be in 3 years?

$P(t) = P_0 a^t = P_0 (1+r)^t = 600\,000\,(1.021)^t$
so $P(3) = 600\,000\,(1.021)^3 \simeq 638\,600$

ii, what is the population of the city using base e ?

then $P(t) = P_0 e^{kt} = P_0 (1+r)^t \Rightarrow e^k = 1+r$

so $e^k = 1.021 \Rightarrow k = \ln(1.021) \simeq 0.0208$

so $P(t) = 600\,000 \; e^{0.0208t}$

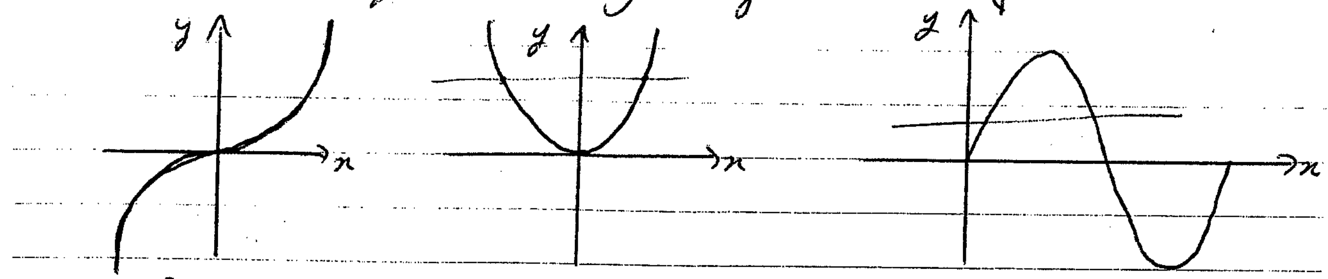
see Ex 1-4 p 34-38

§ 1.6 : Inverse Functions and Logarithms

definition : a function f is called 1-1 if it never takes the same value twice (p 39)

ie if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

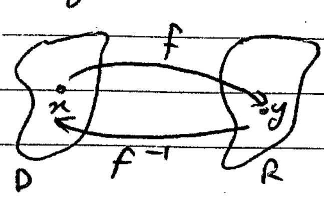
we can test for 1-1 using horizontal lines (p40)



$y = x^3$ 1-1 $\qquad$ $y = x^2$ NOT 1-1 $\qquad$ $y = \sin x$ NOT 1-1

if a function f is 1-1, there is another function, called the inverse of f or inverse function, written f^{-1}, that undoes what f does (p 40)

ie if $f(x) = y$, then $f^{-1}(y) = x$



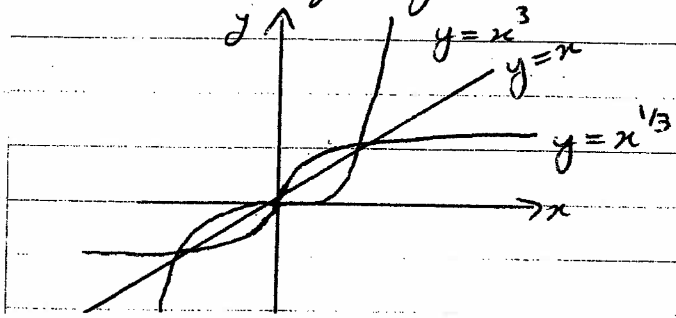
NOTE : $\underset{\text{(inverse)}}{f^{-1}} \neq \frac{1}{f}$ (reciprocal)

and the domain of f^{-1} is the range of f (p40)

" range " f^{-1} " " domain " f

(12)

the inverse of $y = x^3$ is $y = x^{1/3}$



graphically f and f^{-1} are reflections of each other in line $y = x$ (p.41)

given $y = f(x)$, how do we find f^{-1}?
solve for x, swap variables ~~up~~

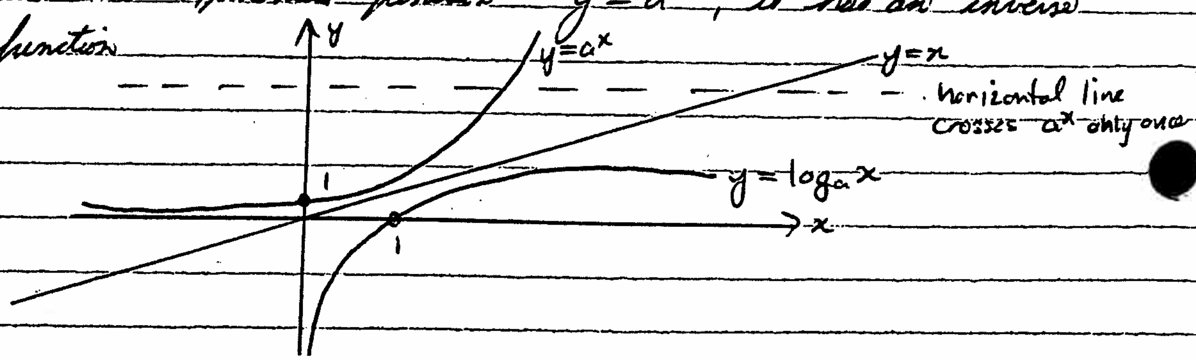
$y = x^3 = f(x)$
so $x = y^{1/3}$
$\therefore y = f^{-1}(x) = x^{1/3}$

also $f^{-1}(f(x)) = x$ (special compositions)
$f(f^{-1}(x)) = x$

$(x^3)^{1/3} = x = (x^{1/3})^3$

see Ex 3 & 4 p 42

consider the exponential function $y = a^x$, it has an inverse function



as we saw ~~in the previous section~~ above, the logarithmic function $y = \log_a x$ $(x>0)$ is defined as the inverse of the exponential $y = a^x$
ie $y = \log_a x$ means $x = a^y$ (p.42)
the most useful bases are 10 and e
the common logarithm $y = \log_{10} x$ is usually written $y = \log x$
the natural logarithm $y = \log_e x$ is written $y = \ln x$ (p.43)

the properties of logs (see page 43 & 44) follow from the properties of exponentials :

i, $\log_a(AB) = \log_a A + \log_a B$

ii, $\log_a(A/B) = \log_a A - \log_a B$

iii, $\log_a(A^p) = p \log_a A$

iv, $\log_a(a^x) = x$ $\qquad e^{\ln x} = x = \ln(e^x)$

v, $a^{\log_a x} = x$

vi, $\log_a 1 = 0$

iii) $f(x) = y = \ln(x+2)$

$e^y = x+2$

$x = e^y - 2$

so $y = f^{-1}(x) = e^x - 2$

examples :

i, solve $\quad 200 = 20e^{3x} \Rightarrow e^{3x} = 10 \Rightarrow 3x = \ln(10) \Rightarrow x \simeq 0.7675$

ii, a picture supposedly painted by Vermeer (1632-1675) contains 99.5% of its carbon 14 is it a fake ?

the half-life of carbon 14 is 5730 years

the amount of carbon 14 present in an object can be modelled by $Q(t) = Q_0 e^{-kt}$

when $t = 5730$ years, $Q(t) = \frac{1}{2}Q_0$, so $e^{-5730k} = 0.5$

then $-5730k = \ln(0.5) \Rightarrow k = \simeq 1.21 \times 10^{-4}$

so $Q(t) = Q_0 e^{-(1.21\times10^{-4})t}$ $\qquad$ so $e^{-(1.21\times 10^{-4})t} = 0.995 \Rightarrow t \approx 41.4$ yrs

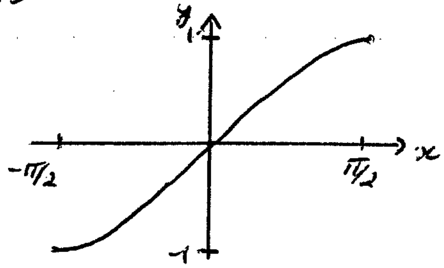
the amount present now is $0.995Q_0$, the painting is fake

see Ex 5-7 p 44-45

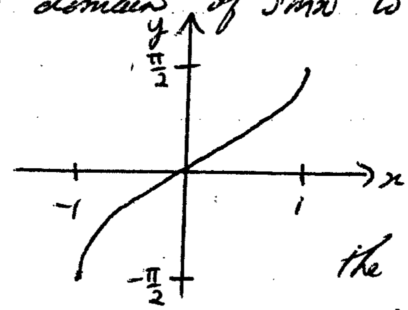
to define the inverse function of $y = \sin x$, we need to remember that the function must pass the horizontal line test

for this reason, we restrict the domain of $\sin x$ to $[-\pi/2, \pi/2]$.

(p 46)



$y = \sin x \quad [-\pi/2, \pi/2]$



the inverse function

$y = \sin^{-1} x = \arcsin x$

(we'll use the notation $\arcsin x$ to avoid notational confusion)

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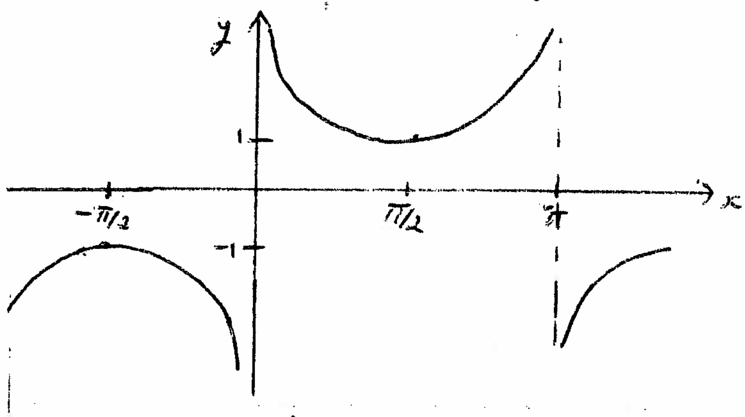
so for $-1 \le x \le 1$ (domain of arcsin x; range of sin y), $y = \arcsin x$ means $x = \sin y$ for $-\pi/2 \le y \le \pi/2$ (domain of sin y; range of arcsin x) (p 47)

it is important to recognize that $\arcsin x$ is <u>not</u> $\csc x = \frac{1}{\sin x}$ (p46)



graph of $y = \csc x$

example: solve $3\sin(2x-3) = 2$

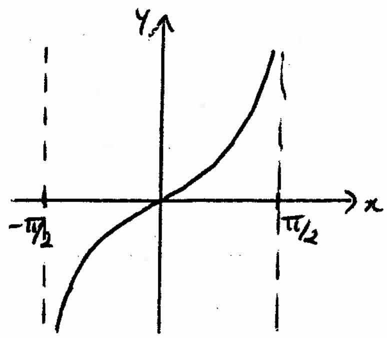
then $\sin(2x-3) = 2/3$

so $2x - 3 = \arcsin(2/3)$

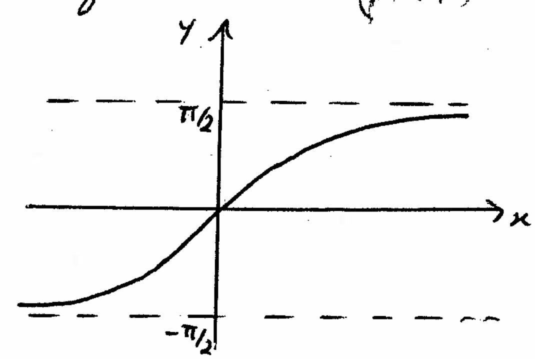
so $2x = \arcsin(2/3) + 3$ and hence

$x = \frac{1}{2}(\arcsin(2/3) + 3) \simeq 1.8649$ (radians)

to define the inverse of the tangent function, we must also restrict the domain of $\tan x$ (p 47)

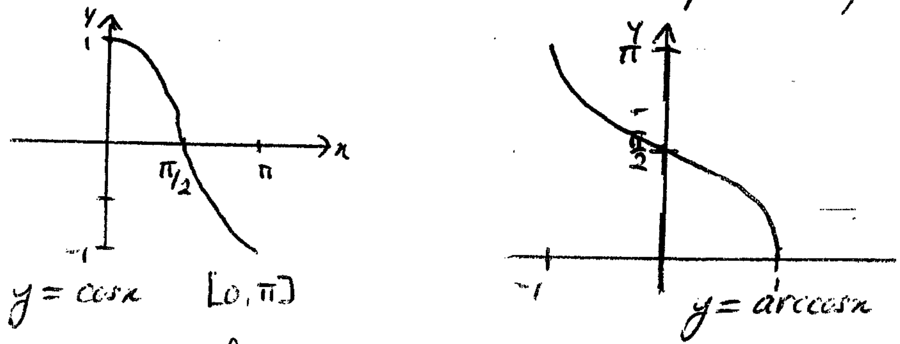


$y = \tan x \quad (-\pi/2, \pi/2)$



$y = \tan^{-1} x = \arctan x$

for any x (domain of arctan x; range of tan y), $y = \arctan x$ means $x = \tan y$ for $-\pi/2 < y < \pi/2$ (domain of tan y; range of arctan x) similarly, we must restrict the domain of $\cos x$ in order to define $\arccos x$: (see #47)



for $-1 \le x \le 1$ (range of $\cos y$; domain of $\arccos x$), $y = \arccos x$ means $x = \cos y$ for $0 \le y \le \pi$ (domain of $\cos y$; range of $\arccos x$)

see pages 46-47 for the definitions of all of the inverse trig functions (though arcsin and arctan are usually the only ones encountered in applications)

see also Ex 8 & 9 p 47-48