

Solutions to the final examination for MATH 1009\* ABCD, Winter 2012.

PART A. MULTIPLE CHOICE QUESTIONS.

Answers: d a c d a d b a c b a c c b c d a

1. The domain of the function  $f(x) = \sqrt{(x+1)(x-4)}$  is

- (a)  $[-1, 4]$ , or equivalently,  $\{x : -1 \leq x \leq 4\}$ .
- (b)  $(-1, 4)$ , or equivalently,  $\{x : -1 < x < 4\}$ .
- (c)  $(-\infty - 1) \cup (4, \infty)$ , or equivalently,  $\{x : x < -1 \text{ and } x > 4\}$ .
- (d)  $(-\infty - 1] \cup [4, \infty)$ , or equivalently,  $\{x : x \leq -1 \text{ and } x \geq 4\}$ .

Answer: d)

2. . What is  $\lim_{x \rightarrow \infty} \frac{x^2 - x + 7}{x^3 + x - 5}$  ?

- a) 0.    b) 1.    c)  $\infty$ .    d) Does not exist.

Answer: a)

3. If  $e^{x+2} = 7$ , what is  $x$  ?

- a)  $\ln 2 - 7$ .                      b)  $7 - \ln 2$ .                      c)  $\ln 7 - 2$ .                      d)  $2 - \ln 7$ .

Answer : c)

4. Which of the following is equal to  $\ln x - \ln x^3 + \ln x^2$  ?

- a)  $\ln(x - x^3 + x^2)$ .                      b)  $6 \ln x$ .                      c)  $2 \ln x$ .                      d) 0.

Answer: d)

5. The expression  $\frac{x^{2.4} \cdot (x^{-0.1})^4}{x^{-3} \cdot (x^2)^2}$  simplifies to

- (a)  $x$ .                      (b)  $x^{-0.5}$ .                      (c)  $x^3$ .                      (d) 1.

Answer: a)

6. Which of the following is equal to  $\log_{\frac{1}{2}} 16$  ?

- a)  $\frac{1}{4}$ .                      b)  $-\frac{1}{4}$ .                      c) 4.                      d)  $-4$ .

Answer: d)

7. What is the slope of the curve  $y = 2^x$  at  $x = 2$ ?

- a)  $2 \ln 2$ .                      b)  $4 \ln 2$ .                      c) 4.                      d) 0.

Answer: b)

8. Given that the cost function for a company is  $C(x) = 60x - 0.025x^2 - 0.01x^3 + 2$  for  $0 < x < 40$ , find the **marginal cost** at a production level of  $x = 20$  units.

- a) 47.                      b) 49.                      c) 55.5.                      d) 1112.

Answer: a)

9. The derivative  $f'$  of  $f(x) = (3x^2 + 1)^{5/3}$  is

- a)  $10(3x^2 + 1)^{8/3}$                       b)  $\frac{5}{3}(3x^2 + 1)^{2/3}$                       c)  $10x(3x + 1)^{2/3}$                       d)  $5x(3x^2 + 1)^{8/3}$

Answer: c)

10. The graph of the function  $y = \frac{3x}{x-1}$  has

- a) no horizontal asymptote and a vertical asymptote  $x = 1$ .  
b) a horizontal asymptote  $y = 3$  and a vertical asymptote  $x = 1$ .  
c) a horizontal asymptote  $y = 3$  and no vertical asymptote.  
d) neither horizontal no vertical asymptote.

Answer: b)

11. What are the critical numbers of the function  $f(x) = 3x^{1/3}$ ?

- a)  $x = 0$ .                      b)  $x = 3$ .                      c)  $x = \frac{1}{3}$ .                      d) none.

Answer: a)

12. A company estimates that its daily cost function is  $C(x) = x^3 - 6x^2 + 13x$  million dollars and its daily revenue function is  $R(x) = 13x$  million dollars. What is the value of  $x$  that maximizes the profit?

- a) 1.                      b) 2.                      c) 4.                      d) 5.

Answer: c)

13. The second derivative  $f''$  of  $f(x) = \log_3(x - 5)$ , ( $x > 5$ ) is

- a)  $\frac{-1}{(x - 5)^2}$                       b)  $\frac{1}{(x - 5)}$                       c)  $\frac{-1}{\ln 3 \cdot (x - 5)^2}$                       d)  $\frac{1}{\ln 3 \cdot (x - 5)}$

Answer: c)

14. Let  $f(x, y) = e^{2x+y}$ . What is  $f_{xx}(0, 1)$ ?

- a)  $e$ .                      b)  $4e$ .                      c) 0.                      d) none of the above.

Answer: b)

15. How long will it take an investment of \$5,000 to grow to \$6,000 if the investment earns interest at the rate of 4% compounded **continuously**?

- a) 9.3years.                      b) 7.2 years.                      c) 4.5 years.                      d) 0.2 years.

Answer: c)

16. Consider the Cobb-Douglas production function  $f(x, y) = 8x^{1/4}y^{3/4}$ , where  $x$  is the number of units of labour and  $y$  is the number of units of capital. What is the marginal productivity of **capital** at  $x = 16$ ,  $y = 81$  ?

- a) 432.                      b)  $\frac{27}{4}$ .                      c) 81.                      d) 4.

Answer: d)

17. The value of  $\int_{-2}^1 (2x + 1) dx$  is

a) 0.

b) 1.

c) 4.

d) 6.

Answer: a)

**PART B. For full marks, be sure to show all of your steps.**

**B1. [ 16 Marks]** Consider the function  $f(x) = x^3 - 3x + 2$ .

[6] (a) Find the intervals where  $f$  is increasing and those where  $f$  is decreasing.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$ . Since  $f' < 0$  for all  $x \in (-1, 1)$ , the function is decreasing for all such  $x$ . Since  $f' > 0$  for all  $x \in (-\infty, -1) \cup (1, \infty)$ , the function is increasing for all such  $x$ .

[3] (b) Find the relative extrema of  $f$  and the values of  $f$  at these points.

$f'(x) = 0 \Rightarrow x = -1, 1, f''(x) = 6x$ ,  
 $f''(-1) < 0 \Rightarrow f(-1) = 4$  is a relative maximum.  
 $f''(1) > 0 \Rightarrow f(1) = 0$  is a relative minimum.

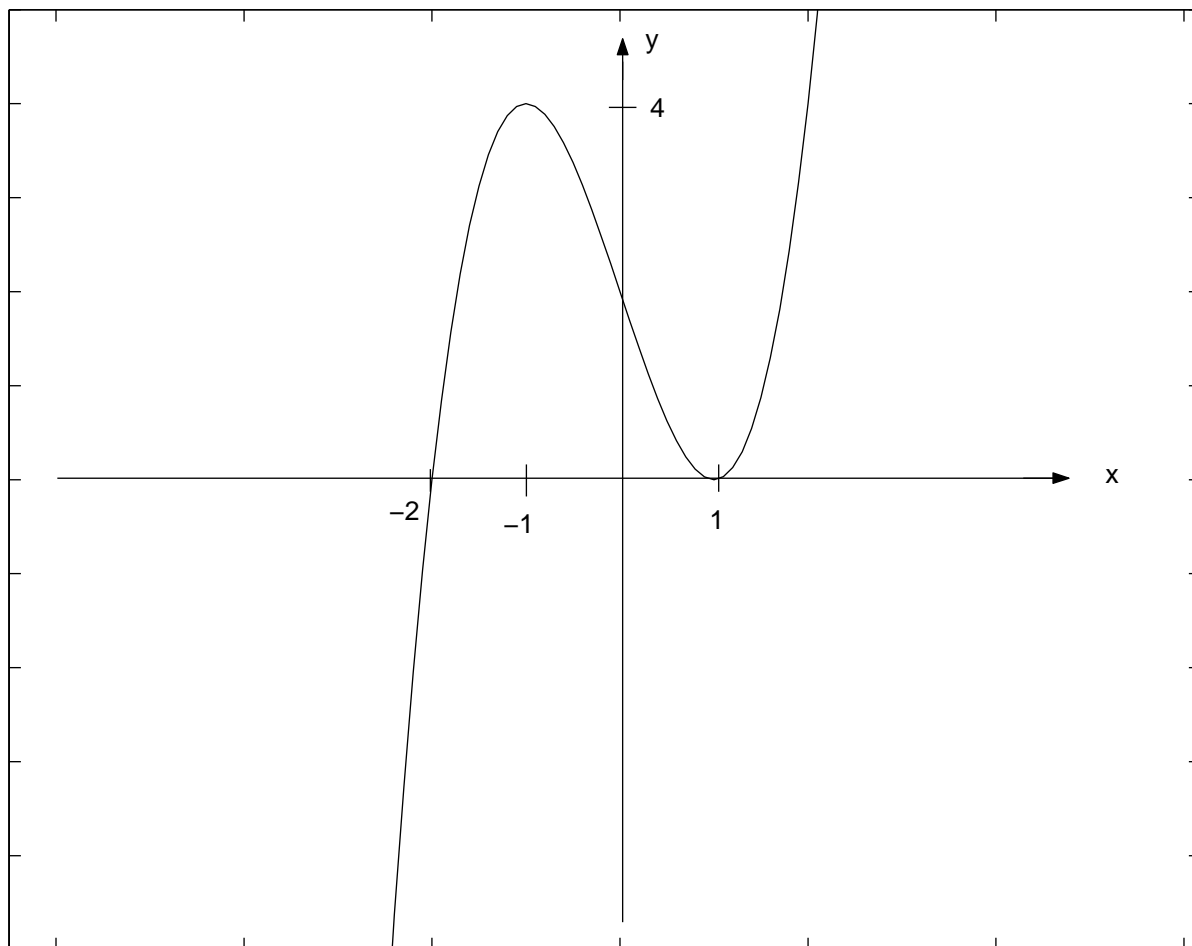
[3] (c) Determine the intervals where the graph of the function is concave up and those where it is concave down.

$f'' = 6x$ .  $f'' > 0$  when  $x > 0$ , so the graph is concave up (CU) for  $x \in (0, \infty)$ .  $f'' < 0$  when  $x < 0$ , so the graph is concave down (CD) for  $x \in (-\infty, 0)$ .

[2] (d) Find the inflection points of  $f$ . (Give both the  $x$ - and the  $y$ -coordinates).

$f'' = 0$  when  $x = 0$ . Since  $f'' = 0$  is changing its sign while moving across  $x = 0$ , therefore  $x = 0$  is an inflection point.  $f(0) = 2$ .

[2] (e) Sketch the graph of  $f$ .



Graph of  $f(x) = x^3 - 3x + 2$

**B2. [10 Marks]**

Use the method of Lagrange multipliers to minimize the function

$$f(x, y) = x^2 + 2y^2 - xy$$

subject to the constraint

$$2x + y = 22.$$

**Solution:**

$$F(x, y, \lambda) = x^2 + 2y^2 - xy + \lambda(2x + y - 22).$$

$$\begin{cases} F_x = 2x - y + 2\lambda = 0, & (1) \\ F_y = 4y - x + \lambda = 0, & (2) \\ F_\lambda = 2x + y - 22 = 0. & (3) \end{cases}$$

From (1)  $\rightarrow 2\lambda = y - 2x$ , so  $\lambda = 0.5y - x$ .

From (2)  $\rightarrow \lambda = x - 4y$ .

Equate  $\lambda$  to obtain  $0.5y - x = x - 4y$ , or  $2x - 4.5y = 0$ . This equation solved together with (3) yields

$$\begin{cases} 2x - 4.5y = 0 \\ 2x + y = 22 \end{cases} \Rightarrow 5.5y = 22, \quad y = 4.$$

Then from (3),  $x = \frac{22 - y}{2} = 9$ . Thus, the constrained minimum is  $f(9, 4) = 77$ .

**B3. [10 Marks]**

[4] (a) Find all the critical points of the function  $f(x, y) = x^2 - 6xy + y^2 - 16y + 10$ . (Do NOT classify them.)

[6] (b) The function  $f(x, y) = 2x^3 + y^2 - 6x + 4y$  has two critical points  $(1, -2)$  and  $(-1, -2)$ . Use the Second Derivative Test to classify each point as local maximum, local minimum or saddle.

**Solution:**

(a)  $f_x = 2x - 6y = 0$ ,  $f_y = -6x + 2y - 16 = 0$ . Solving these two equations for  $x$  and  $y$  yields  $y = -1$  and  $x = -3$ . Thus, the only critical point is  $(-3, -1)$ .

(b) The Second Derivative Test:

$$f_{xx} = 12x, \quad f_{yy} = 2, \quad f_{xy} = 0, \quad \text{so } D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 12x \cdot 2 = 24x.$$

$D(1, -2) = 24 > 0$ ,  $f_{xx}(1, -2) = 12 > 0$ , therefore  $(1, -2)$  is a point of a local minimum.

$D(-1, -2) = -24 < 0$ , therefore  $(-1, -2)$  is a saddle point.

**B4. [13 Marks]**

(a) Evaluate the following integrals:

$$[2] \text{ (i) } \int \left( 5x^4 + \frac{1}{x} \right) dx = 5 \int x^4 dx + \int \frac{1}{x} dx = 5 \cdot \frac{x^5}{5} + \ln|x| + C = x^5 + \ln|x| + C.$$

$$[4] \text{ (ii) } \int_0^1 (3x - 2)^4 dx$$

Let  $u = 3x - 2$ , then  $\frac{du}{dx} = 3$ ,  $dx = \frac{du}{3}$ .

Now change the limits of integration:  $x = 0 \Rightarrow u = 3 \cdot 0 - 2 = -2$ ,  $x = 1 \Rightarrow u = 3 \cdot 1 - 2 = 1$ .  
Thus,

$$\int_0^1 (3x - 2)^4 dx = \int_{-2}^1 u^4 \frac{du}{3} = \frac{1}{3} \int_{-2}^1 u^4 du = \frac{1}{3} \cdot \frac{u^5}{5} \Big|_{-2}^1 = \frac{1}{15} [1^5 - (-2)^5] = \frac{33}{15}.$$

$$[3] \text{ (iii) } \int 6x^2 e^{x^3+1} dx = \int 6x^2 e^u \left( \frac{du}{3x^2} \right) = 2 \int e^u du = 2e^u + C = 2e^{x^3+1} + C$$

[4] **(b)** Consider the equation  $xy + y^2 - x^2 = 10$ , where  $y = y(x)$  is defined implicitly as a function of  $x$ . Find  $y' = \frac{dy}{dx}$ .

Rewrite expression, stating that  $y = y(x)$  is a function of  $x$ :

$$x \cdot y(x) + (y(x))^2 - x^2 = 10.$$

Differentiate both parts of the above expression with respect to  $x$ :

$$1 \cdot y(x) + x \cdot y'(x) + 2y \cdot y'(x) - 2x = 0.$$

Solve for  $y'(x)$ :

$$x \cdot y'(x) + 2y(x) \cdot y'(x) = 2x - y(x), \quad y'(x)(x + 2y) = 2x - y, \quad y'(x) = \frac{2x - y}{x + 2y}.$$