Summary individual pages

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Summary Part 1 (Ch.1): Classification.

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- PDE: Has a partial derivative in it.
  - Ex: $u_{xx}+u_{yy}+u_{xy}=0$ (2^{nd} order)
- ODE: Has only ordinary derivatives in it.
  - Ex: $y'+y=0$ (1^{st} order)
- Order: The order of the highest derivative in the DE is called the order of the DE
  - Ex: $y''+x'=0$; is of the second order.
- Linearity: Must be able to be reduced to the general form

$$y' + p(x)y = g(x) \quad or \quad \frac{dy}{dx} + p(x)y = g(x)$$

  - Red flags:
    - The function cannot be raised to any power (other than 1)
      - Ex: (y^2 is not linear)
    - The function or the derivative(s) (y,y'' etc) cannot be inside another function.
      - Ex: (cos(y) or cos(y'') are not linear)
    - The function cannot be multiplied by one of its derivatives.
      - Ex: (yy' is not linear)
      - Ex: $y'''+yy''+2y^2+\sin y''=0$
        - Because of all three red flags.
- Homogeneous: Meaning $y=y_c$
  - x^2+y^2+xy is homogeneous because it is equal to $(tx)^2+(ty)^2+txty= t^2(x^2+y^2+xy)$
  - so is $x=\sqrt{x^2+y^2}$ because that => $x^2=x^2+y^2$
  - Also the equation must =0.

Summary Part 2: (Ch.2) First order ODE's part a.

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1. Check for separation:
   a. Needs form $\frac{dy}{dx} = g(x)\mathrm{h}(y)$ to solve this:
      i. Integrate all h(y) with respect to y, all g(x) with respect to x, usually with the assumption that y≠0 for existence.
      Note: no need for multiple c (constants of integration), one will do.
      ii. Isolate for y. Then check case of y=0 on the newfound function and include it in the interval if its correct.
      iii. If IVP, plug in values and solve for c.
2. Check if linear:
   a. Needs to be in the form $\frac{dy}{dx} + p(x)y = g(x)$ so check for 3 red flags.
      i. Knowing p(x) from the above formula, find $e^{\int p(x)dx}$
      ii. $e^{\int p(x)dx} * y = \int \left(e^{\int p(x)dx} * g(x)dx\right)$ then integrate (keep a c).
      iii. Isolate for y so that you have an expression $y = y_c + y_p$ where y_c is the function that has the constant c (and maybe x) in it and y_p is the function that doesn't. This gives you the explicit one parameter family of solutions for y
      iv. If IVP, plug in values to find c and that's your implicit solution.
3. Check if exact:
   a. Needs to be in the form $M(x,y)dx + N(x,y)dy = 0$
      i. This is exact if $\partial y M(x,y) = \partial x N(x,y)$.
      - If exact:
        - Integrate with respect to x or y to find f(x,y):
          $\int M(x,y)dx = f(x,y)$ or $\int N(x,y)dy = f(x,y)$.
        - Derive with respect to y or x to find c. Note that after the derivation, you will have c'(y) and c'(x) respectively and will need to solve for it.
          $\partial y \int M(x,y)dx = \partial y f(x,y) = N(x,y)$
          or
          $\partial x \int N(x,y)dy = \partial x f(x,y) = M(x,y)$.
        - This will give you something like:
          $$\partial y \int M(x,y)dx = N(x,y) = 4x + c'(y) = 4x - 8y^3$$
          - Then integrate c'(y) and $8y^3$ and find your c(y) value which may be a function +c.
        - If IVP, plug in values of x and y to solve for c(y).
- If not exact:
  - Solve either $\frac{\partial yM - \partial xN}{N}$ (resulting function solely of x) or $\frac{\partial yM - \partial xN}{M}$ (resulting function solely of y), take integrating factor $e^{\int resulting\ function\ dx}$ (or dy)
  - Multiply M(x,y)+N(x,y)=0 by the integrating factor.
    i. Check again to see if $\partial y M(x,y) = \partial x N(x,y)$. If it does, then solve the same as an exact equation. If not, you may have to take another integrating factor or check your work.

Summary Part 3: (Ch.2) First order ODE's part b:

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1. Check for substitution: (if IVP, you can convert initial value from y(x) to u(x) simply by plugging it into the substitution. Usually more complicated though. y(x)= to u(x)=; if $u=y^2$, y(0)=6, then u(0)=36.
   a. Substitution of y=ux gives dy=udx+xdu (use for exact form) or y'=u'x+u (use for linear form).
      i. Replace y and y' with your substitutions.
      ii. Solve using separation, linear, exact etc.
      iii. Plug y back into final answer.
      iv. If IVP, solve for c.
   b. If of the form $\frac{dy}{dx} + p(x)y = g(x)y^n$ (Bernoulli):
      i. Identify P(x), g(x) and n.
      ii. Substitution of $u=y^{1-n}$ gives new linear equation $\frac{du}{dx} + (1-n)P(x)u = (1-n)g(x)$ then solve like a linear equation.
      iii. Substitute $u=y^{(1-n)}$ back into the equation and isolate for y to solve.
      iv. If IVP, solve for as if you were doing a linear equation.
   c. Substitution of u=Ax+By+c gives (if $u=2x+y^2+20$) u'=2+2yy' (chain rule).
      i. Replace y and y' with your substitutions.
      ii. Solve using separation, linear, exact etc.
      iii. Plug y back into final answer.
      iv. If IVP, solve for c.

Summary Part 4: (Ch.3) Second order ODE's part a.

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General equation: $ay''+by'+cy=g(x)$, $a\neq 0$

Linearly dependent or independent:

For $y_c=c_1y_1+c_2y_2$:

Wrouskian: $W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y'_2(x)-y'_1(x)y_2(x) \neq 0$ for any x, then linearly independent. This would mean $y_c=c_1y_1+c_2y_2$ is a general solution of $ay''+by'+cy=0$.

Fundamental sets:

$\{y_1,y_2\}$ is a fundamental set if y_1 and y_2 are linearly independent.

Note: your y_p cannot be part of the fundamental set.

Ex: $\{e^x,xe^x\}$ y_p would be x^2e^x.

<u>Homogeneous y_c:</u>

$ay''+by'+cy=0$

Gives a general solution of:

$y_c=c_1y_1+c_2y_2$ for any $c_1,c_2 \in \mathbb{C}$

Three cases for auxiliary equation of y_c: $ak^2+bk+c=0$ (*)

1) If $b^2-4ac>0$ then $y_1(x)=e^{k1x}$ and $y_2(x)=e^{k2x}$, where $k_1\neq k_2$ and k_1 and k_2 are real solutions to (*).
   a) Fund. Set $\{e^{k1x},e^{k2x}\}$ so $y_c=c_1e^{k1x}+c_2e^{k2x}$

2) If $b^2-4ac=0$ then $y_1(x)=e^{k1x}$ and $y_2(x)=xe^{k2x}$, where $k_1=k_2=-\frac{b}{2a}\in\mathbb{R}$ is a repeated root of (*).
   a) Fund. Set $\{e^{k1x},xe^{k2x}\}$ so $y_c=c_1e^{k1x}+c_2xe^{k2x}$

3) If $b^2-4ac<0$ then $y_1(x)=e^{k1x}$ and $y_2(x)=e^{k2x}$, where $k_1=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $k_2=\overline{k_1}$. (of the form $k_1=5-i$, $k_2=5+i$).
   a) Fund. Set $\{e^{k1x},e^{k2x}\}$ so $y_c=c_1e^{k1x}+c_2e^{k2x}$ or $\{Rey_1,Imy_1\}$ so if $y_1=e^{3+5ix}$, $y_2=e^{3-5ix}$ then $y_c=e^{3x}[c_1\cos(5x)+c_2\sin(5x)]$

      $y_c=e^{(Re)x}[c_1\cos(Imx)+c_2\sin(Imx)]$ --> $\{e^{(Re+iIm)x},e^{(Re-iIm)x}\}$

ex: $8e^{-5ix}=8(\cos(-5x)+i\sin(-5x))$

To find k's, either $(k-1)(k+4)$ gives $k=1,-4$ or use quadratic formula.

To solve:

1) y_c: $ay''+by'+cy=0$
   a) Auxiliary equation = $ak^2+bk+c=0$
      i) Using three cases: k_1 and k_2 =____
      ii) Find y_1 and y_2 and the fundamental set.
   b) If IVP ,$y(0)$=___, $y'(0)$=___ ;

      $y_c=c_1y_1+c_2y_2$

      y'_c=_________

      Solve for c_1 and c_2 by plugging in IVP and having two equations and two unknowns.

Summary Part 5: (Ch.3) Second order ODE's part b.

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Non-Homogeneous y_p:

Equation: $ay''+by'+cy=g(x)$, $g(x)\neq 0$

Gives a general solution of:

$y=y_c+y_p=c_1y_1+c_2y_2+y_p$ for any $c_1,c_2 \in \mathbb{C}$

Method of guessing for y_p: (doesn't work if there's ln or tan)

Guess for a y_p value then solve for y'_p and y''_p that satisfies the equation $ay''+by'+cy=g(x)$.

Guess a $y_p(x)$, then find $y_p'(x)$ and $y_p''(x)$ and then plug it in.

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x+B\sin 4x$
6. $\cos 4x$	$A\cos 4x+B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

To find the values of A, B, C etc, either:

1) Use y, y' and y'' to solve for three constants using linear algebra. Coefficients before like terms are the same.

   a) Ex: if $y''-2y'+y= 2\cos x+x^2-1$,

Coefficients of	cosx	=2
Coefficients of	Sinx	=0
Coefficients of	x^2	=1
Coefficients of	x	=0
Coefficients of	constant	=-1

   b) $y_p= A\cos x+B\sin x+Cx^2+Dx+E$.

   $y_p'=-A\sin x+B\cos x+2Cx+D$

   $y_p''=-A\cos x-B\sin x+2C$

   Which gives (when plugged in):

   $(-A\cos x-B\sin x+2C)+(2A\sin x-2B\cos x-4Cx-2D)+(A\cos x+B\sin x+Cx^2+Dx+E)= 2\cos x+x^2-1$

   c) Coefficients of cosx: $-A-2B+A=2$

   Coefficients of sinx: $-B+2A+B=0$

   Coefficients of x^2: $C=1$

   Coefficients of x: $4C+D=0$

   Coefficients of constant: $2C-2D+E=-1$

Note: no part of your y_p may be part of the fundamental set of solutions otherwise it will always be a solution.

Ex: $y''-2y'+y=4+e^x$;

From y_c you have fund. set: $\{e^x,xe^x\}$

Now your y_p would normally be $4+Ae^x$ but now it can't because e^x is in the fund set. To solve this simply multiply your common term by x.

Note: in this example xe^x is also in the fund set so you need to multiply twice to get $y_p=4+Ax^2e^x$.

Now you would solve for y_p' and y_p''.

Note: when checking coefficients of ____, note that x^2e^x is one term and xe^x is another.

Summary Part 6: (Ch.3) Second order ODE's part c.

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Variation of parameters for finding y_p:

$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$ || variation of parameter c_1 and c_2 from y_c

Where : $ay_p''+by_p'+cy_p=g(x)$	!!!!!!(Note: 'a' must be 1)!!!!!! If it isn't, divide everything by a.

$$u_1=\int\frac{\begin{vmatrix}0 & y_2\\ g(x) & y'_2\end{vmatrix}dx}{W}\ \ and\ u_2=\int\frac{\begin{vmatrix}y_1 & 0\\ y'_1 & g(x)\end{vmatrix}dx}{W}$$

Steps:

1) Find $y_c=c_1y_1+c_2y_2$
2) Then $y_p=u_1y_1+u_2y_2$
   a) $W=\begin{vmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}=$ ______ (Note: if this =0 then it is linearly dependent and can't be solved)
   b) $W_1=\begin{vmatrix}0 & y_2\\ g(x) & y'_2\end{vmatrix}=$ ______
   c) $W_2=\begin{vmatrix}y_1 & 0\\ y'_1 & g(x)\end{vmatrix}=$ ______
   d) Now find u_1 and u_2 using the formulas $u_1=\int\frac{W_1}{W}$ $and\ u_2=\int\frac{W_2}{W}$
3) $y=c_1y_1+c_2y_2+u_1y_1+u_2y_2$

Cauchy-Euler:

$ax^2y''+bxy'+cy=g(x)$ => note new x part of equation

Change of variable or substitution:

$t=\ln x$ or $x=e^t$

Gives:

$$a\ddot{y}+(b-a)\dot{y}+cy=g(e^t)$$

Auxiliary equation:

$ak^2+(b-a)k+c=0$

Theorem:

Case 1: If the discriminant, $(b-a)^2-4ac>0$, => $k_1\in\mathbb{R}$, $k_2\in\mathbb{R}$, $k_1\neq k_2$

i)

$\{y_1(x),y_2(x)\}$	$=\{x^{k1},x^{k2}\}$
$\{y_1(t),y_2(t)\}$	$=\{e^{k1t},e^{k2t}\}$

Case 2: If the discriminant, $(b-a)^2-4ac=0$, => $k_1=k_2\in\mathbb{R}$

ii)

$\{y_1(x),y_2(x)\}$	$=\{x^{k1},x^{k1}\ln x\}$
$\{y_1(t),y_2(t)\}$	$=\{e^{k1t},te^{k1t}\}$

Case 3: If the discriminant, $(b-a)^2-4ac<0$, => $k_1=\overline{k_2}$, $k_1\in\mathbb{R}$

iii)

$\{y_1(x),y_2(x)\}$	$=\{e^{k1\ln x},e^{k2\ln x}\}$ (complex)	$\{x^{REk1}\cos(Imk_1\ln x),x^{REk1}\sin(Imk_1\ln x)\}$ (REal)
$\{y_1(t),y_2(t)\}$	$=\{e^{k1t},e^{k2t}\}$ (complex)	$\{e^{(REk1)t}\cos(Imk_1t),e^{(REk1)t}\sin(Imk_1t)\}$ (REal)

Ex: $k_1=5+7i$ => $\{x^5\cos(7\ln x),x^5\sin(7\ln x)\}$ => $\{e^{5t}\cos(7t),e^{5t}\sin(7t)\}$

Summary Part 7: (Ch.3) Second order ODE's part d.

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<u>Non-Linear:</u>
General second order non linear ODE:

F(x,y,y',y'')=0

We consider:

u(y)=y'	F(y,y',y'')=0	(no x)
u(x)=>y'	F(x,y',y'')=0	(no y)

Either way, substitute u=y' and solve the resulting 1st order equation for u

Ex: $x^2y''+(y')^2=0$, $(y')^2$ is not linear.

a) Since no y, take u(x)=y' which gives u(x)'=y''
b) Now substitute:

   $x^2u'+u^2=0$ => $x^2\frac{du}{dx}+u^2=0$
c) Now solve using separation. (keep the +c from the integration)

   $$\frac{du}{u^2}=-\frac{dx}{x^2}$$
   i) Check if u=0 plugged in works.
d) You end up with u=some f(x) with a constant c. Swap u back with y' and integrate to find y.
   i) Now you will have another constant of integration A so for your final answer write y=____, c,A∈ℝ

Ex: $y^2y''=y'$

a) Since no x, u(y)=y', giving $u'(y)=y''=\frac{du}{dy}\frac{dy}{dx}=u\frac{du}{dy}$
b) Now substitute:

   $y^2u\frac{du}{dy}=u$
c) Separation of variables.
d) Resubstitute y' back in for u

Etc:

Summary Part 8: (Ch.3) Linear Models.

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Linear Models:

Springs:

Period of oscillations:

$$T = \left(\frac{2\pi}{\sqrt{\frac{k}{m}}}\right)$$

Where:

x(t_0)=x(t_0+T)

x'(t_0)=x'(t_0+T)

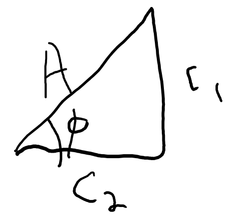
Frequency:

$$f = \frac{1}{T} = \left(\frac{\sqrt{\frac{k}{m}}}{2\pi}\right)$$

Angular frequency:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

$\omega^2 = \frac{K}{m}$



Undamped Springs:

$$F = kx\ (hooke's\ law)$$

$$F = ma = m\frac{d^2x}{dt^2} = mx''$$

$$mx'' = -kx$$

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{v_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \Rightarrow \textit{Equation of motion}$$

$Y = A\sin(\omega t + \phi)$

$A\cos(\omega t - \phi)$

Damped Springs:

General equation:

$$mx'' + \beta x' + kx = 0$$

Where: k= Spring Constant, β= Damping Constant

Other forms:

$$x'' + \frac{\beta}{m}x' + \frac{k}{m}x = x'' + 2\lambda x' + \omega^2 x = 0$$

Steps:

Auxiliary equation: $n^2 + \frac{\beta}{m}n + \frac{k}{m} = 0$

$n = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

3 cases :

1) $n_1 \neq n_2$ or $\frac{\beta^2}{m^2} - 4\omega^2 > 0$ or $\beta > 2\omega m$ and $n_1, n_2 \in \mathbb{R}$,=> Over Damped. No Oscillation.

$$x(t) = C_1e^{n_1t} + C_2e^{n_2t} = C_1e^{\left(\frac{-\beta}{2m} + \sqrt{\frac{\beta^2}{4m^2}-\omega^2}\right)t} + C_2e^{\left(\frac{-\beta}{2m} - \sqrt{\frac{\beta^2}{4m^2}-\omega^2}\right)t}$$

<0 <0

2) $n_1 = n_2$ or $\frac{\beta^2}{m^2} - 4\omega^2 = 0$ or $\beta = 2\omega m$ and $n_1 = n_2 = \frac{-\beta}{2m}$,=> Critically Damped. No Oscillation.

$$x(t) = C_1e^{n_1t} + tC_1e^{n_1t} = C_1e^{\left(\frac{-\beta}{2m}\right)t} + tC_1e^{\left(\frac{-\beta}{2m}\right)t}$$

3) $n_2 = \overline{n_1}$ or $\frac{\beta^2}{m^2} - 4\omega^2 < 0$ or $\beta < 2\omega m$ and $n_1 = n_2 = \in \mathbb{C}$, => Under Damped. Yes Oscillation.

$$x(t) = C_1e^{n_1t} + C_2e^{\overline{n_1}t} = C_1e^{\left(\frac{-\beta}{2m}\right)t}e^{i\left(\sqrt{\omega^2-\frac{\beta^2}{4m^2}}\right)t} + C_2e^{\left(\frac{-\beta}{2m}\right)t}e^{-i\left(\sqrt{\omega^2-\frac{\beta^2}{4m^2}}\right)t}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

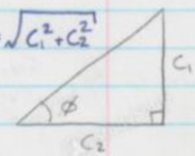
overdamped if $R^2 - 4L/C > 0,$

critically damped if $R^2 - 4L/C = 0,$

underdamped if $R^2 - 4L/C < 0.$

Alternative form of $x(t)$: This form is important because it gives the actual amplitude of free vibrations. The harmonic motion eq seen prior does give the intial displacement, although the amplitude of free vibration will be slightly larger. Simple harmonic series eq. can be changed into a shifted sine or cosine function of the form:

$A = \sqrt{c_1^2 + c_2^2}$



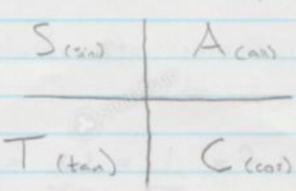
shifted sine function

$$y = A\sin(\omega t + \phi)$$

A = amplitude
ω = angular frequency
ϕ = phase shift

$$\left.\begin{aligned} \sin\phi &= \frac{c_1}{A} \\ \cos\phi &= \frac{c_2}{A} \end{aligned}\right\} \quad \phi = \arctan\left(\frac{c_1}{c_2}\right)$$

NOTE: In computing the phase angle (ϕ), the signs of the coeff.'s c_1 & c_2 must correlate with the sin and cos functions. This is done using the CAST system. The graph below assigns each trig function to a given quadrant along with that labelled A (A = all). This graph tells you where the particular trig func. you are looking at is positive and negative. All three func.'s are positive in A. For example. sin is only positive in the 1st and 2nd quadrant, were as cos is only positive in the 1st and 4th quadrant.



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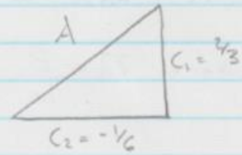
for simplicity, we affiliate C_1 with sin and C_2 with cos. Hence once the phase angle is first calculated, it must correlate with the signs of (C_1 and sin) & (C_2 and cos).

<u>Ex:</u> $C_1 = 2/3$, $C_2 = -1/6$, $\omega = 8$ } want to put in the form of a shifted sin function.

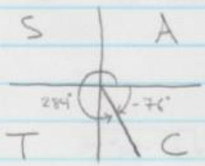
→ $y = A \sin(\omega t + \phi)$



$A = \sqrt{(2/3)^2 + (-1/6)^2} \approx 0.687$

$\phi = \arctan\left(\frac{C_1}{C_2}\right) \rightarrow \arctan\left(\frac{2/3}{-1/6}\right) = \arctan(-4) = -1.325\text{rad}$

convert rad → degrees: $-1.325 \cdot \frac{180}{\pi} \approx -76^\circ$ or 284°



This is the wrong phase angle since it does not agree with the fact that sin is positive, correlated with $C_1 = 2/3$. Also, cos is positive in this quadrant, which goes against our sign convention since $C_2 = -1/6$. Hence we shift the phase angle by π (ie $-1.325 + \pi$ or $-76^\circ + 180 = 104^\circ$). Now the phase angle is located in the 2^{nd} quadrant, agreeing with our sign convention. Hence the true shifted sin function is:

$$y = 0.687(8t + 1.816)$$

<u>NOTE</u>: A similar (but not exact) methode can be used for the shifted cosine function. I did not see this in class but it is covered in the textbook.

Free Damped Motion (ie: fluid resistance)

→ damping force (F_d) is considered to be proportional to the instantanious velocity (dx/dt) multiplied by some power (β). where is say β is the damping constant.

DE: $x'' + 2\lambda x + \omega^2 x = 0$ or $mx'' + \beta x' + kx = 0$

where $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$

→ with free damped motion, we have 3 cases that arise from the auxiliary equation: $r^2 + 2\lambda r + \omega^2 = 0$. These are the same as we had seen in section 2 of the class.

Case #1: $\lambda^2 - \omega^2 > 0$ then the system is overdamped, and the solution will have the form of:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$ or

$$x(t) = e^{-\lambda t}\left(C_1 e^{\sqrt{-\lambda^2-\omega^2}\,t} + C_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

Case #2: $\lambda^2 - \omega^2 = 0$ then the system is critically damped, and the solution will be in the form:

$$x(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$$

Case #3: $\lambda^2 - \omega^2 < 0$ then the system is said to be underdamped, and r_1 & r_2 will be complex numbers:

$$r_1 = -\lambda + \sqrt{\omega^2-\lambda^2}\,i \quad , \quad r_2 = -\lambda + \sqrt{\omega^2-\lambda^2}\,i$$

general solution → $x(t) = e^{-\lambda t}\left(C_1 \cos(\sqrt{\omega^2-\lambda^2}\,t) + C_2 \sin(\sqrt{\omega^2-\lambda^2}\,t)\right)$

Spring mass System + Driven force:

previously we have 2nd order ODEs that were homogeneous. IE, in the form $mx'' + Bx' + Kx = 0$. Now we are adding into the equation an external/driving force, denoted $f(t)$. Equation now becomes:

$$m x'' + Bx' + Kx = f(x)$$

→ The corresponding solution to the above will be in the form $y = y_c + y_p$. where y_c is the solution to the homogeneous equation (ie: $f(t) = 0$); and y_p is any particular equation satisfying the nonhomogeneous eq. (ie: $f(t) \neq 0$)

* Once again, methodes for solving for y_p have been covered in section 2 of the class (refer to last review). Methodes will include ① variation of parameters and ② methode of undetermined coefficients.

LRC - Series electrical circuit

Here, all equations and procedures are the same with only the exception of a change of variables.

$E(t)$ → voltage (ie: external force)
L → inductance
C → capacitance
R → resistance
$\hat{i} = \hat{i}(t)$ → current

* $\hat{i} = \frac{dq}{dt}$ = $q(t)$ = electric charge in capacitor

} formulas for current and charge on next page →

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Current ($\hat{i}(t)$): $L\hat{i}'(t) + R\hat{i} + \frac{1}{C}q = E(t)$

or equivalently,

$$L\frac{d\hat{i}}{dt} + R\hat{i} + \frac{1}{C}q = E(t)$$

charge ($q(t)$): $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

or

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

*For change of variables, we are dealing with the form below in each case:

$$ay'' + by' + cy = g(x)$$

→ auxiliary eq is always $ar^2 + br + c = 0$ (remember that $a=1$ to use quadratic f.)

→ discriminant: $b^2 - 4ac$

→ for LRC: $a = L$, $b = R$, $c = \frac{1}{C}$, $g = E$; for Mass-Spring: $a = m$, $b = \beta$, $c = k$, $g = f$

NOTE For LRC-electric series circuit, the DE for charge is obtained from deriving the current formula. Substitute $i = \frac{dq}{dt}$ in current formula and see that you get current formula. Hence we can alter between solutions by taking the derivative or integration of one of the above depending on what we want to compute.

Summary part 8: (Ch.1.3,2.7) Models part a.

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Population/Radioactive growth and decay:

$\frac{dp}{dt} = kp$ where:

- k is some coefficient/constant of proportionality and p it the population itself.
- p(t) = total population at time t
- p'(t) or $\frac{dp}{dt}(t)$ = rate of change of the population
- k>0 for growth and k<0 for decay.
- $p(0)=p_0$ is the initial condition.

Solve using separation of variables.

Gives: $p(t) = p_0e^{kt}$

Notes:

- Growth or decay rate must be proportional to its size.
- Radioactive is always decay.

Newtons law od cooling/warming:

$\frac{dT}{df} = k(T - T_m)$ where:

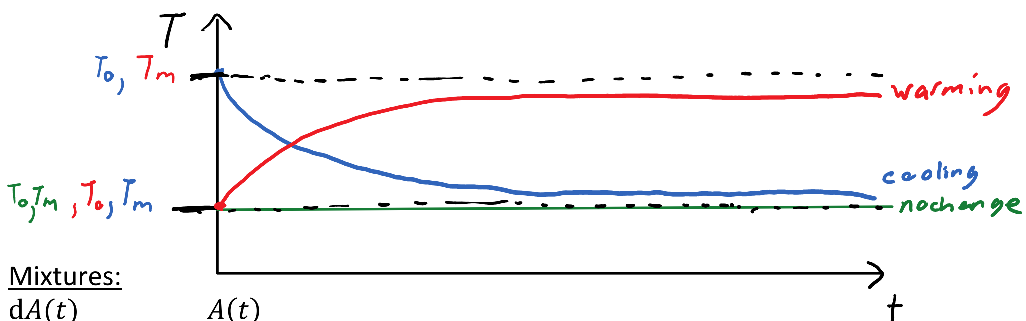
- T(t) = temperature of body at time t.
- T_m= temperature of surrounding medium.
- $\frac{dT}{dt}(t)$= rate of change of the temperature.
- $\frac{dT}{dt}$ = k(T-T_m) where k is a constant of proportionailty again.
- $T(0) = T_0$ is the initial condition.

Solve using separation of variables.

Gives: $T(t) = T_m + (T_0 - T_m)e^{kt}$

Notes:

$T_0 > T_m$ =Cooling.
$T_0 < T_m$ = Warming.
$T_0 = T_m$ = Constant temp.



Mixtures:

$$\frac{dA(t)}{dt} = C_{in}v - \frac{A(t)}{V}v$$

Where:

A(t) is amount of solute in solvent.
C is the concentration in mass/liter.
C_{in}*v is the input rate of solute.
$\frac{A(t)}{V}v$ is the output rate of solute.
V is the original solution in liters.
v is the solution being pumped in in liters.

Solving gives:

$$A(t) = (A_0 - C_{in}V)e^{\frac{-v}{V}t} + C_{in}V$$

Notes:

if $(A_0 - C_{in}V)$ >0, gives exponential decay towards C_{in}*V
if $(A_0 - C_{in}V)$ <0, gives exponential growth towards C_{in}*V

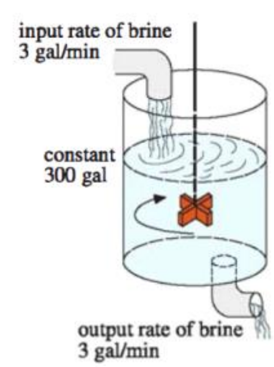


FIGURE 1.3.3 Mixing tank

Summary part 9: (Ch.1.3,2.7) Models part b.

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Spread of Disease:

$$\frac{dx}{dt} = kxy$$

Where:

x(t) is infected people.

y(t) is not yet infected people.

K is the constant of proportionality. and

xy is proportional to the number of interactions.

n is the population of people

Solving gives:

$$\frac{dx}{dt} = kx(n + 1 - x)$$

Notes:

Initial equation here is x(0)=1 , meaning it started with one infected person.

Chemical Reactions:

Gives equation:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = k(\alpha - x)(\beta - x)$$

Where:

X(t) is the amount of substance A remaining at any point in time.

K is negative constant of proportionality since this is decay.

x is the amount of product formed.

α and β are the given amounts of the two reagents.

$(\alpha - x)$ and $(\beta - x)$ are the instantaneous amounts of reagent not yet turned into product.

Draining a Tank:

Gives equation:

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$

Where:

g is acceleration due to gravity.

A_h is the area of the hole through which the water is draining.

h is the depth of water remaining in the tank.

V(t) is the volume of water in the tank at time t.

\- sign indicates the volume is decreasing.

Or:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Where:

A_w is the surface area of the water.

$\frac{\mathrm{d}h}{\mathrm{d}t}$ is the height of water at time t.

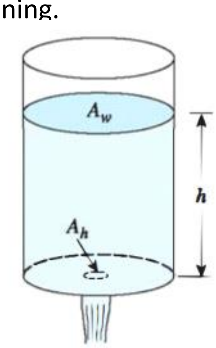

FIGURE 1.3.4 Water draining from a tank

Summary part 10: (Ch.1.3,2.7) Models part c.

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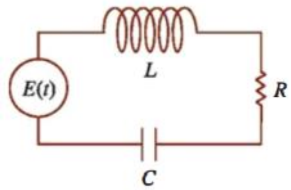
Series Circuits:

Solving gives :

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{C}q=E(t)$$



(a) LRC-series circuit

Where:

- i=q(t) is the charge at time t.
- L is inductance.
- R is resistance.
- C is capacitance.
  - All three are generally constants.
- E(t) is the voltage in a circuit.

Falling Bodies (with air resistance):

Solving gives:

$$m\frac{\mathrm{d}v}{\mathrm{d}t}=mg-kv$$

Where:

- v(t) is the velocity with respect to time.
- m is the mass.
- g is force of gravity.
- K is the drag coefficient (if no air resistance k=0).
- $\frac{\mathrm{d}v}{\mathrm{d}t}$ is acceleration.

And:

$$m\frac{\mathrm{d}^2s}{\mathrm{d}t^2}+k\frac{\mathrm{d}s}{dt}=mg$$

Where:

- s(t) is distance the body has fallen with respect to time.

Suspended Cables:

Solving gives:

$$\frac{\mathrm{d}y}{\mathrm{dx}}=\frac{W}{T_1}$$

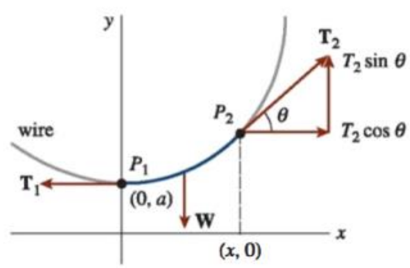


FIGURE 1.3.9 Element of cable

Where:

- $\frac{\mathrm{d}y}{\mathrm{dx}}$ is the slope of the wire, could be anything.
- W is the vertical load which $=T_2\sin\theta$.
- T_1 is the tangential force to the cable. $=T_2\cos\theta$

Summary Part 11 (Ch.10): Liner systems.

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Form:

$$\vec{y}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{y} + \vec{f}(x)$$

Steps:

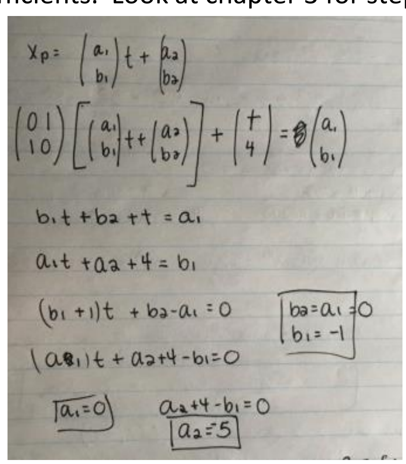
1) Find y_c:
   a. Characteristic polynomial and eigen values using:
      $\lambda^2 - (\text{Tr}A)\lambda + \det A = 0$
   b. 3 cases:
      $\lambda_1 \neq \lambda_2$:
      I. $(A-\lambda_1 I)\begin{bmatrix} 1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\overrightarrow{\phi_1}$
      II. $(A-\lambda_2 I)\begin{bmatrix} 1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\overrightarrow{\phi_2}$
      III. $\overrightarrow{y_c} = C_1\overrightarrow{\phi_1} \cdot e^{\lambda_1 x} + C_2 \cdot \overrightarrow{\phi_2} e^{\lambda_2 x}$ where $c_1, c_2 \epsilon \mathbb{C}$

      $\lambda_1 = \lambda_2$:
      i. A is diagonal: $A = \begin{bmatrix} a & 0 \\ O & d \end{bmatrix}$ then $a=d=\lambda_1=\lambda_2$.
         I. $\overrightarrow{\phi_1}$ and $\overrightarrow{\phi_2}$ are any two linearly independent vectors, easy to take $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
         II. $\overrightarrow{y_c} = C_1\overrightarrow{\phi_1} \cdot e^{\lambda x} + C_2 \cdot \overrightarrow{\phi_2} e^{\lambda x}$ where $c_1, c_2 \epsilon \mathbb{C}$

      ii. A is <u>not</u> diagonal:
         I. $(A-\lambda I)\begin{bmatrix} 1 \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $\vec{\phi}$
         II. $(A-\lambda I)\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$ gived $\vec{\varphi}$
         III. $\overrightarrow{y_c} = C_1\vec{\phi} \cdot e^{\lambda x} + C_2 \cdot \left(\vec{\phi} x e^{\lambda x} + \vec{\varphi} e^{\lambda x}\right)$ where $c_1, c_2 \epsilon \mathbb{C}$

   Note: If $\lambda = x+iy$, $\overrightarrow{\phi_2}$ = the conjugate of $\overrightarrow{\phi_1}$, ex: $\overrightarrow{\phi_1}, \overrightarrow{\phi_2} = \begin{bmatrix} 1 \\ 2+3i \end{bmatrix}, \begin{bmatrix} 1 \\ 2-3i \end{bmatrix}$

2) Find y_p:
   a. Educated guess and method of undetermined coefficients. Look at chapter 3 for steps. Apply those steps to each part of the matrix.
   b. Variation of parameters:
      i. $\vec{Y} = [\overrightarrow{y_1}\,\overrightarrow{y_2}] = \begin{bmatrix} y_{1a} & y_{2a} \\ y_{1b} & y_{2b} \end{bmatrix}$
      ii. $\vec{Y}^{-1} = \dfrac{1}{detY}\begin{bmatrix} y_{2b} & -y_{2a} \\ -y_{1b} & y_{1a} \end{bmatrix}$
      iii. $\overrightarrow{y_p} = \vec{Y} * \int \vec{Y}^{-1} \vec{f} dx$



3) $\vec{y} = \overrightarrow{y_c} + \overrightarrow{y_p} = \vec{Y}\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \int \vec{Y}^{-1} \vec{f} dx\right)$

IVP: $\vec{y}(0) = \begin{bmatrix} q \\ r \end{bmatrix}$:

1) Ex: $\overrightarrow{y_c}(0) = C_1 * \_\_\_ + C_2 * \_\_\_ = \begin{bmatrix} q \\ r \end{bmatrix}$ and plug in x=0 and then solve for c_1 and c_2 using the matrix.

Notes chapter 10:

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Note:

Finding the real component of an answer:

Answer = $\vec{y}(x) = \frac{2+i}{2}\begin{bmatrix}1\\2-i\end{bmatrix}e^{(4+i)x} + \frac{2-i}{2}\begin{bmatrix}1\\2+i\end{bmatrix}e^{(4-i)x}$

In real values:

$$\vec{y}(x) = e^{4x}\left(\begin{bmatrix}\frac{2+i}{2}\\ \frac{5}{2}\end{bmatrix}(cosx + isinx) + \begin{bmatrix}\frac{2-i}{2}\\ \frac{5}{2}\end{bmatrix}(\cos x - i\sin x)\right)$$

1st component: $\left(\frac{2+i}{2} + \frac{2-i}{2}\right)\cos x + \left(\frac{2i-1}{2} + \frac{-2i+1}{2}\right)\sin x = 2\cos x - \sin x$

2nd component: $\left(\frac{5}{2} + \frac{5}{2}\right)\cos x + \left(i\frac{5}{2} - i\frac{5}{2}\right)\sin x = 5\cos x$

Total = $\overrightarrow{y_{(x)}} = e^{4x}\begin{bmatrix}2\cos x - \sin x\\ 5\cos x\end{bmatrix}$

Note: Algebraic multiplicity:

$\lambda^2 - 7\lambda + 10 = 0$

Gives λ_1=2, λ_2=5.

This is of algebraic multiplicity 1 because each λ (2 and 5) are only there once.

$(\lambda-2)^2=0$

Gives λ=2.

This is of algebraic multiplicity 2 because the same λ is in the characteristic polynomial twice.

Summary Part 12 (Ch.5) : Power Series Representation.

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Homogeneous:

ay''+P(x)y'+Q(x)y=0

1) Plug the following into the above equation:

$$\begin{cases} y = \sum_{n=0}^{\infty} C_n x^n \\ y' = \sum_{n=1}^{\infty} C_n (n) x^{n-1} \\ y'' = \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} \end{cases}$$

Gives:

$$a \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} + P(x) \sum_{n=1}^{\infty} C_n (n) x^{n-1} + Q(x) \sum_{n=0}^{\infty} C_n x^n$$

2) Make all the x's be of the same power j:

$$a \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} + P(x) \sum_{n=1}^{\infty} C_n (n) x^{n-1} + Q(x) \sum_{n=0}^{\infty} C_n x^n$$

j=n-2 j+2=n	j=n-1 j+1=n	j=n n=j

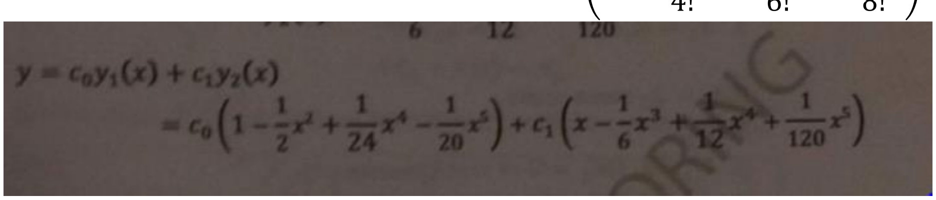
$$a \sum_{j=0}^{\infty} C_{j+2}(j+2)(j+1) x^j + P(x) \sum_{j=0}^{\infty} C_{j+1}(j+1) x^j + Q(x) \sum_{j=0}^{\infty} C_j x^j$$

3) Make all the sums start at the same j=( ) and combine them all:
   This would be done by taking the first values of the sums out and writing them before.

$$\_\_\_\_\_\_\_ + \sum_{j=0}^{\infty} \left[a \left(C_{j+2}(j+2)(j+1) \right) + P(x) \left(C_{j+1}(j+1) \right) + Q(x) \left(C_j \right) \right] x^j$$

4) Make everything before the x^j =0 and everything in front of the summation =0 and isolate for the highest constant. This could give a couple equations.
   a. All values that have only constants (ex: $2C_2+C_0=0$) will be one equation.
   b. All values that have x (ex: $5xC_3+C_1=0$) will be another equation.
   c. All values that have x^2 (ex: $17x^2C_4+C_2=0$) will be a third equation etc.
   d. Everything that is multiplied by x_j is another equation.
      i. Here isolate for $C_{(j+2)}$ and make a table of values for different values of j to find the rest of the constants. Make sure to write all $C_{( )}$'s in terms of C_0 and C_1. On the final, they will ask for the first 3-4 NON-ZERO constants. To find these:
         1) coefficients of $y_1(x)$ plug in $\Rightarrow C_1 = 0, C_0 = C_0$
         2) coefficients of $y_2(x)$ plug in $\Rightarrow C_1 = C_1, C_0 = 0$

5) Forms of solution:
   a. variations of the following : $C_{2k} = (-1)^k \left(\frac{C_0}{(2k)!} \right), \quad C_{2k+1} = \left(\frac{2^k C_1}{(2k+1)!} \right)$
   b. variations of the following: $y = C_0 \left(2 + \frac{4^2 x^2}{4!} + \frac{4^3 x^4}{6!} + \frac{4^4 x^6}{8!} \right) + C_1 \left(\frac{4x}{3!} + \frac{4^2 x^3}{5!} + \frac{4^3 x^5}{7!} \right)$



$$y = c_0 y_1(x) + c_1 y_2(x) = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5\right)$$

Notes chapter 5:

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Remember that if you have:

5x $\sum_{n=1}^{\infty} C_n(n)x^{n-1}$

That equals

$$\sum_{n=1}^{\infty} 5C_n(n)x^{n-1+1} = \sum_{n=1}^{\infty} 5C_n(n)x^{n}$$