University of Ottawa
Dept. of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 3x - Spring-Summer 2015
Midterm II -[redacted]
Professor: Abdelkrim El basraoui
Duration: 80 minutes.

Name: Solution ID Number: ________________

Instructions: (Please read carefully.)

- This exam has 8 pages and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Read each question carefully before answering.
- **All questions** are long answer questions and they are worth 5 marks each. **A correct answer requires a full, clearly-written and detailed solution.**
- Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Throughout this test, any reference to the **first octant** in 3-dimensional space means the region $x \geq 0$, $y \geq 0$, $z \geq 0$.
- Do not unstaple the test.
- Good luck!

Question	Q.1	Q.2	Q.3	Q.4	Q.5	Total
Maximum	5	5	5	5	5 + 2 bonus pts	25
Score						

1. Consider the solid in the first octant bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x = 2$, $y = 1$ and the surface $z = 6 + xy$. This solid has a mass density given by the function $\delta(x, y, z) = 2x + y$. Find the total mass of this solid.

- The solid is given by

$$E = \{(x,y,z) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 1,\ 0 \leq z \leq 6+xy\}$$

- The mass is then

$$m = \iiint_E \delta(x,y,z)\, dV = \int_0^2 \int_0^1 \int_0^{6+xy} (2x+y)\, dz\, dy\, dx$$

$$m = \int_0^2 \int_0^1 (2x+y)(6+xy)\, dy\, dx$$

$$= \int_0^2 \int_0^1 \left(12x + 2x^2y + 6y + xy^2\right) dy\, dx$$

$$= \int_0^2 \left[12xy + x^2y^2 + 3y^2 + \frac{1}{3}xy^3\right]_0^1 dx$$

$$= \int_0^2 \left(\frac{37}{3}x + x^2 + 3\right) dx$$

$$= \left[\frac{37}{6}x^2 + \frac{x^3}{3} + 3x\right]_0^2$$

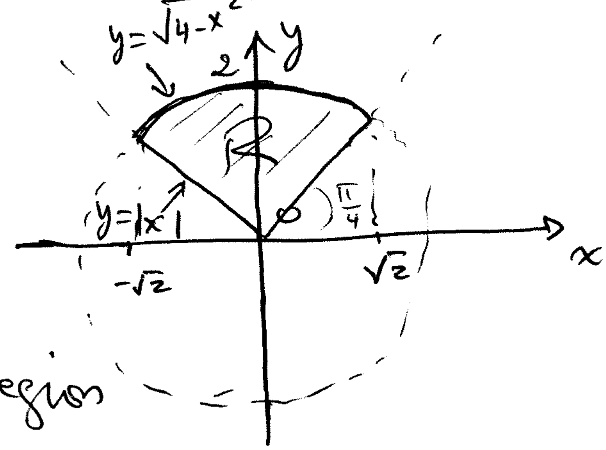
$$\boxed{m = \frac{100}{3}}$$

2. Compute the following double integral

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{|x|}^{\sqrt{4-x^2}} \sqrt{(x^2+y^2)}\, dy\, dx$$

Hint: Sketch the region of integration as a type I region in the x-y plane, and then express this region and the integral in a different coordinate system.

- Region: $R = \{(x,y) \mid |x| \le \sqrt{2},\ |x| \le y \le \sqrt{4-x^2}\}$



- In polar coordinates the region is $\{(r,\theta) \mid 0 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$.

- The double integral becomes then

$$I = \int_0^2 \int_{\pi/4}^{3\pi/4} r \quad r\, d\theta\, dr \qquad (\sqrt{x^2+y^2} = \sqrt{r^2} = r)$$

$$= \frac{\pi}{2} \int_0^2 r^2\, dr = \frac{\pi}{2} \left[\frac{r^3}{3}\right]_0^2$$

$$\boxed{I = \frac{4\pi}{3}}$$

3. Consider the solid in the first octant which is bounded by the planes $z = 0$, $z = 2$, $y = 0$, $x = 0$ and the surfaces $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. This solid has a mass density given by the function $\delta(x, y, z) = zx^2 + zy^2$. Set up a triple integral in **cyclindrical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

- In cylindrical coordinates the solid is given by

$$E = \left\{ (r, \theta, z) \mid 1 \leq r \leq 2 \ , \ 0 \leq \theta \leq \frac{\pi}{2} \ , \ 0 \leq z \leq 2 \right\}$$

- Also $\delta(x, y, z) = \delta(r\cos(\theta), r\sin(\theta), z) = zr^2$.

- The mass is therefore

$$m = \int_1^2 \int_0^{\pi/2} \int_0^2 zr^2 \quad r \, dz \, d\theta \, dr$$

$$\boxed{m = \int_1^2 \int_0^{\pi/2} \int_0^2 zr^3 \, dz \, d\theta \, dr}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = 4t\,\vec{\mathbf{i}} + 2\cos(3t)\,\vec{\mathbf{j}} + 2\sin(3t)\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/6.$$

- $\vec{r}\,'(t) = 4\,\vec{i} - 6\sin(3t)\,\vec{j} + 6\cos(3t)\,\vec{k}$.
- $\|\vec{r}\,'(t)\| = \sqrt{16 + (-6\sin(3t))^2 + (6\cos(3t))^2}$

$$= \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$$

- Arclength :

$$L = \int_0^{\pi/6} \|\vec{r}\,'(t)\|\,dt = 2\sqrt{13}\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{13}\,\pi}{3}}$$

5. Consider the surface which is the portion of the sphere $x^2 + y^2 + z^2 = 1$ that lies between the planes $z = -\frac{\sqrt{2}}{2}$ and $z = \frac{\sqrt{2}}{2}$. (See figures on the next page).

(a) Give a parametrization of this surface.

(b) Set up an integral which would give the area of this surface, **but DO NOT evaluate this integral**.

(c) **BONUS [2 marks]** Evaluate the integral in (b). Note that you are eligible to receive bonus marks **only if you have the correct answer in (b).**

(a) • In sperical coordinates

$$\begin{cases} x = \sin(\phi)\cos(\theta) \\ y = \sin(\phi)\sin(\theta) \\ z = \cos(\phi) \end{cases} ; \quad \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi .$$

$$\therefore \vec{r}(\phi,\theta) = \langle \underbrace{\sin(\phi)\cos(\theta)}_{x}, \underbrace{\sin(\phi)\sin(\theta)}_{y}, \underbrace{\cos(\phi)}_{z} \rangle$$

$$\frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi$$

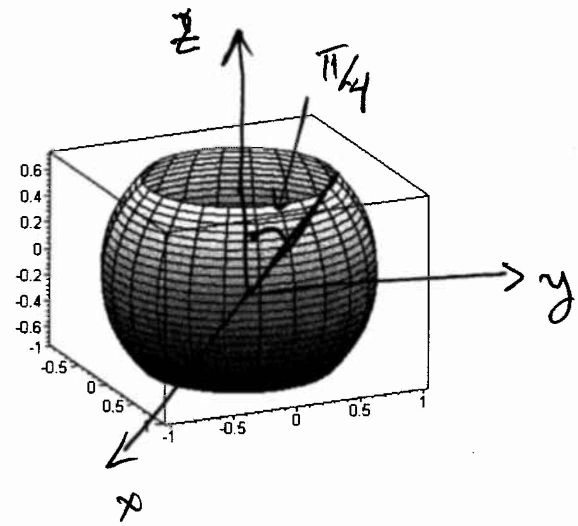
(b) • Surface area: $S = \iint_D \|\vec{r}_\phi \times \vec{r}_\theta\| \, dA$

where $D = \{(\phi,\theta) \mid \frac{\pi}{4} \leq \phi \leq \frac{3\pi}{4}, \ 0 \leq \theta \leq 2\pi\}$.

$$\vec{r}_\phi = \langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi) \rangle$$
$$\vec{r}_\theta = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0 \rangle$$

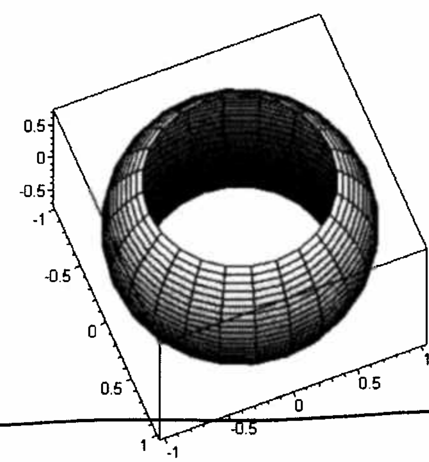
$$\therefore \vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \rangle$$

Two different views of the surface described in problem 5.



& we have

$$\|\vec{r}_\phi \times \vec{r}_\theta\| = \sin(\phi).$$



• Thus $$\boxed{S = \iint_D \sin(\phi)\, dA = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \sin(\phi)\, d\phi\, d\theta}$$

(c) Evaluation: $$S = 2\pi \int_{\pi/4}^{3\pi/4} \sin(\phi)\, d\phi = 2\pi \Big[-\cos(\phi)\Big]_{\pi/4}^{3\pi/4}$$

$$\therefore \boxed{S = 2\pi\sqrt{2}}$$