WILFRID LAURIER UNIVERSITY
WATERLOO, ONTARIO

Fall Term, 2001

Course ID: ***MA103***

Course Title: ***Calculus I***

Name: Solutions

Student ID: ____________

Section: ________

Professors: ***S. Bulman-Fleming,*** (Section A); ***K. Cameron,*** (Section B); ***E. Wang,*** (Section C); ***D. Harmsworth,*** (Section D); ***D. Harmsworth,*** (Section F)

Number of Pages: ***8 plus cover page***

Length of Examination: ***2 hours***

Examination Aids Allowed: **Calculators are permitted. No other aids are allowed.**

INSTRUCTIONS:

Please place your ID card on the right-hand side of the desk.

You are not allowed to leave the examination room until **one hour** *after the start of the exam and you* **must sign the identification sheet before leaving.**

Your examination paper must be handed in at the front of the exam room before you leave. No paper of any kind is to be taken from the room. When you are finished please leave the exam room quietly.

Cheating on an examination will result in an "F" grade in the course concerned and possible suspension from the University.

All notes, briefcases and books must be deposited at the front of the room unless you are writing an open book examination.

You are not allowed to use your own paper for rough work. If you need scrap paper, raise your hand and a proctor will supply with you with some.

If you must leave the room for personal reasons, please raise your hand and a proctor will escort you.

Make sure you are sitting at the correct paper. Count the pages to be certain that no pages are missing.

Do not begin this examination until you are instructed to do so.

Total Value: *80 marks*

Answer in the spaces provided, using backs of pages for additional space if necessary.

Show all your work. Insufficient justification will result in a loss of marks.

The formulas on page 8 are provided for your convenience.

[4 marks] 1. (a) State F.T.O.C. (I) (the Fundamental Theorem of Calculus, part I.)

If f is a continuous function on $[a,b]$ (1)

then where g is the function defined by $g(x) = \int_a^x f(t)\,dt$, (2)

(g is continuous on $[a,b]$, g is differentiable on (a,b),

and) $g'(x) = f(x)$ for all $x \in (a,b)$. (1)

Alternative: If f is continuous (1) on $[a,b]$, then

$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$. (3)

[7 marks] (b) Use (a) and differentiation to show that

$$\int_0^{\sin x} \sqrt{1-t^2}\,dt = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) \text{ for } 0 \le x \le \frac{\pi}{2}.$$

[**Hint:** Do **NOT** attempt to evaluate the integral.]

$$\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}\,dt = \sqrt{1-\sin^2 x} \cdot \cos x \quad (1)\ (1)$$

$$= \cos^2 x$$

$$\frac{d}{dx}\left(\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right) = \frac{1}{2}(1 + \cos 2x) \quad (1)$$

$$= \cos^2 x$$

equality (1)

$$\therefore \int_0^{\sin x} \sqrt{1-t^2}\,dt = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C \quad (1)$$

To determine C, use a convenient value of x, $x = 0$.

$$\int_0^0 \sqrt{1-t^2}\,dt = 0 = 0 + C \quad (1)$$

$$\therefore C = 0 \quad (1)$$

$\therefore$ Given equation holds.

[5 marks] 2. Evaluate $\int \sec^3 x \tan^5 x \, dx$. [NOT COVERED FALL, 2002]

[**Hint:** First write $\sec^3 x \tan^5 x$ as $\sec^2 x \tan^4 x \sec x \tan x$.]

$$\int \sec^3 x \tan^5 x \, dx$$

$$= \int \sec^2 x \, (\tan^2 x)^2 \sec x \tan x \, dx$$

$$= \int \sec^2 x \, (\sec^2 x - 1)^2 \sec x \tan x \, dx \quad (1)$$

Let $u = \sec x$
$du = \sec x \tan x \, dx$

$$= \int u^2 (u^2 - 1)^2 \, du \quad (1)$$

$$= \int (u^6 - 2u^4 + u^2) \, du \quad (1)$$

$$= \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C \quad (1)$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C \quad (1)$$

[9 marks] 3. Evaluate $\int \frac{\sqrt{9x^2 - 4}}{x} dx$ by using the trigonometric substitution $3x = 2\sec\theta$.

Let $3x = 2\sec\theta$ ie. $\theta = \sec^{-1}\left(\frac{3x}{2}\right)$ $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$$3\,dx = 2\sec\theta\tan\theta \, d\theta \quad (1)$$

$$\sqrt{9x^2 - 4} = \sqrt{4\sec^2 x - 4} = 2\sqrt{\sec^2\theta - 1} = 2|\tan\theta| = 2\tan\theta \quad (1)$$

$$I = \int \frac{2\tan\theta}{2/3 \sec\theta} \frac{2}{3}\sec\theta\tan\theta \, d\theta \quad (1)$$

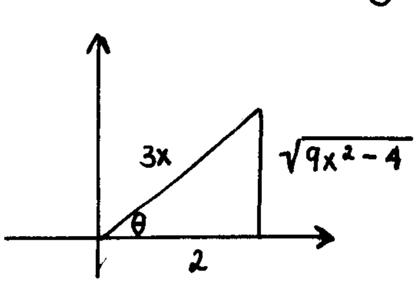
$$= 2\int \tan^2\theta \, d\theta \quad (1)$$

$$= 2\int (\sec^2\theta - 1) \, d\theta \quad (1)$$

$$= 2(\tan\theta - \theta) + C \quad (2)$$

$$= 2\left(\frac{\sqrt{9x^2 - 4}}{2} - \sec^{-1}\left(\frac{3x}{2}\right)\right) + C$$

(1) (1)



(-1) if no C on this page

[3 marks] 4. (a) Use L'Hospital's Rule to evaluate $\lim_{t\to\infty} e^{-4t}(4t+1)$. $= L$ IF $0 \cdot \infty$

$$L = \lim_{t\to\infty} \frac{4t+1}{e^{4t}} \quad \text{IF } \frac{\infty}{\infty} \quad (1)$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{4}{4e^{4t}} \quad (1)$$

$$= 0 \quad (1)$$

[8 marks] (b) Evaluate the improper integral $\int_0^\infty xe^{-4x}\,dx$ $= I$ or show that it is divergent.

[**Hint**: You may find the answer to (a) helpful.]

$$I = \lim_{t\to\infty} \int_0^t x e^{-4x}\,dx \quad (1)$$

(2) $u = x$, $du = dx$; $dv = e^{-4x}\,dx$, $v = -\frac{1}{4}e^{-4x}$

$$= \lim_{t\to\infty} \left[-\frac{xe^{-4x}}{4}\Big|_0^t + \frac{1}{4}\int_0^t e^{-4x}\,dx \right]$$

(1) IBP formula

$$= \lim_{t\to\infty} \left[\frac{-te^{-4t}}{4} - \frac{e^{-4x}}{16}\Big|_0^t \right] \quad (1)$$

$$= \lim_{t\to\infty} \left[\frac{-te^{-4t}}{4} - \frac{e^{-4t}}{16} + \frac{1}{16} \right]$$

(1) handling evaluation at t and 0 correctly

$$= \lim_{t\to\infty} -\frac{1}{16}\left[(4t+1)e^{-4t} - 1 \right]$$

(1) putting in form to use (a)

$$= \frac{1}{16} \quad (1)$$

[6 marks] 5. Find the area bounded by the curve $y = x^2\sqrt{x+1}$ and the line $y = -1$ from $x = 0$ to $x = 3$.

$$A = \int_0^3 \left(x^2\sqrt{x+1} + 1\right) dx \quad (1)$$

Let $u = x+1$ (1)

$du = dx$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 4$

$$= \int_1^4 \left((u-1)^2 u^{1/2} + 1\right) du \quad (1)$$

$$= \int_1^4 \left(u^{5/2} - 2u^{3/2} + u^{1/2} + 1\right) du \quad (1)$$

$$= \left(\frac{2}{7} u^{7/2} - 2\cdot\frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + u\right)_1^4 \quad (1)$$

$$= \frac{2}{7}(128-1) - \frac{4}{5}(32-1) + \frac{2}{3}(8-1) + (4-1)$$

$$= 19\frac{16}{105} \quad (1) \qquad \frac{1900+95+16}{105} = \boxed{\frac{2011}{105}}$$

(or 19.1524)

[5 marks] 6. Suppose that $z = f\left(2s-t, \frac{t}{2} - s\right)$. Show that $\frac{\partial z}{\partial s} + 2\frac{\partial z}{\partial t} = 0$.

[**Hint:** Set $x = 2s - t$ and $y = \frac{t}{2} - s$. Then $z = f(x, y)$ and a Chain Rule may be used.]

$z = f(x, y) \qquad x = 2s - t \qquad y = \frac{t}{2} - s$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad (1)$$

$$= \frac{\partial z}{\partial x}(2) + \frac{\partial z}{\partial y}(-1) \quad (1)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \quad (1)$$

$$= \frac{\partial z}{\partial x}(-1) + \frac{\partial z}{\partial y}\left(\frac{1}{2}\right) \quad (1)$$

$$\frac{\partial z}{\partial s} + 2\frac{\partial z}{\partial t} = 2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + 2\left(-\frac{\partial z}{\partial x} + \frac{1}{2}\frac{\partial z}{\partial y}\right)$$

$$= 0 \quad (1)$$

Alternative Solution :

[6 marks] 5. Find the area bounded by the curve $y = x^2\sqrt{x+1}$ and the line $y = -1$ from $x = 0$ to $x = 3$.

$$A = \int_0^3 \left(x^2\sqrt{x+1} + 1\right) dx$$

$$= \int_1^2 \left((u^2-1)^2 u + 1\right) 2u\,du$$

$$= 2\int_1^2 \left(u^6 - 2u^4 + u^2 + u\right) du$$

$$= 2\left(\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + \frac{u^2}{2}\right)\Bigg|_1^2$$

$$= 2\left(\frac{127}{7} - \frac{2\cdot 31}{5} + \frac{7}{3} + \frac{3}{2}\right)$$

$$= 19\,\frac{16}{105}$$

Let $u = \sqrt{x+1}$
$u^2 = x+1$
$u^2 - 1 = x$
$2u\,du = dx$
$x = 0 \Rightarrow u = 1$
$x = 3 \Rightarrow u = 2$

[5 marks] 6. Suppose that $z = f\left(2s - t, \frac{t}{2} - s\right)$. Show that $\frac{\partial z}{\partial s} + 2\frac{\partial z}{\partial t} = 0$.

[**Hint:** Set $x = 2s - t$ and $y = \frac{t}{2} - s$. Then $z = f(x, y)$ and a Chain Rule may be used.]

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}(2) + \frac{\partial z}{\partial y}(-1) \quad (2)$$

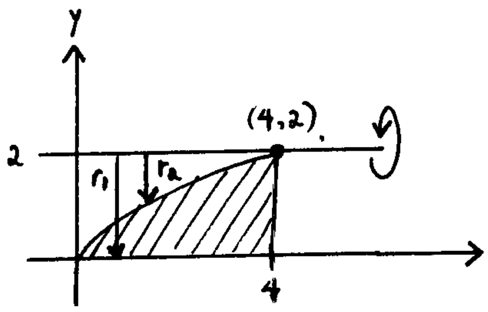
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}(-1) + \frac{\partial z}{\partial y}\left(\frac{1}{2}\right) \quad (2)$$

$$\therefore \frac{\partial z}{\partial s} + 2\frac{\partial z}{\partial t} = 0.$$

(1)

[NOT COVERED, FALL 2002]

[*8 marks*] 7. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the line $y = 2$.



r_1 = outer radius = 2 ①

r_2 = inner radius = $2 - y = 2 - \sqrt{x}$ ①

$$V = \pi \int_0^4 \left(2^2 - (2-\sqrt{x})^2\right) dx \quad ①\ ①\ ①$$

$$= \pi \int_0^4 (4\sqrt{x} - x)\, dx \quad ①$$

$$= \pi \left(\frac{4 \cdot 2x^{3/2}}{3} - \frac{x^2}{2} \right) \Big|_0^4 \quad ①$$

$$= \pi \left(\frac{64}{3} - 8 \right) \quad ①$$

$$= \frac{40\pi}{3}$$

"Global scheme"
- ① for π
- ① for limits of int.
- ③ for integrand
- ② for antiderivative
- ① for numerical work

[*5 marks*] 8. Show that the function $u = \ln\sqrt{x^2+y^2}$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \, 2x \ ① = \frac{x}{x^2+y^2} \quad ①$$

$$u_{xx} = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \ ① = \frac{y^2-x^2}{(x^2+y^2)^2} \quad ①$$

By symmetry, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ ①

$$\therefore \quad u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

It's somewhat easier to write $u = \frac{1}{2}\ln(x^2+y^2)$ first. If they do this, give ① mark, then ① more for $u_x = \dfrac{x}{x^2+y^2}$.

[10 marks] 9. Evaluate $\int \frac{9x^2+3x+13}{(x-2)(x^2+2x+3)}\,dx$ using partial fractions.

$$\frac{9x^2+3x+13}{(x-2)(x^2+2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+3} \qquad (1)$$

$$9x^2+3x+13 = A(x^2+2x+3) + (Bx+C)(x-2) \qquad (1)$$

$$= (A+B)x^2 + (2A-2B+C)x + (3A-2C)$$

$x = 2 \Rightarrow 55 = 11A \quad \therefore A = 5$

$A+B = 9 \quad \therefore B = 4$ (2)

$3A-2C = 13 \quad \therefore C = 1$

(1) Set-up
(1) Clear denoms
(2) Solve
(1) $\ln_1$
(3) $\ln_2$
(2) $\tan^{-1}$

$$I = \int\left(\frac{5}{x-2} + \frac{4x+1}{x^2+2x+3}\right)dx$$

$$= 5\ln|x-2| + \int\frac{4x+1}{(x+1)^2+2}\,dx \qquad (1)\text{ complete square}$$

$$= 5\ln|x-2| + 2\int\frac{2x+2}{(x+1)^2+2}\,dx - 3\int\frac{dx}{(x+1)^2+2} \qquad (2)\text{ split + borrow}$$

$$= 5\ln|x-2| + 2\ln|(x+1)^2+2| - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C$$

(1) (1) (1)

Alternative after completing square:

Let $u = x+1$, $du = dx$ $\qquad 4x+1 = 4(u-1)+1 = 4u-3$ (1) substitute

$$\int\frac{4x+1}{(x+1)^2+2}\,dx = \int\frac{4u-3}{u^2+2}\,du$$

$$= 2\int\frac{2u}{u^2+2}\,du - 3\int\frac{du}{u^2+2} \qquad (1)\text{ split}$$

$$= 2\ln|u^2+2| - \frac{3}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C$$

$$= 2\ln|(x+1)^2+2| - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + C$$

[*10 marks*]10. Find all critical points and determine the relative extrema (if there are any) of $f(x, y) = x^3 - 3xy + 3y^2 - 9y$. Use the Second Derivative Test to justify your conclusion.

$f_x = 3x^2 - 3y$ ①

$f_y = -3x + 6y - 9$ ①

$f_x = 0 \Leftrightarrow x^2 = y$

$f_y = 0 \Leftrightarrow -x + 2y - 3 = 0$

$2x^2 - x - 3 = 0$

$(2x-3)(x+1) = 0$

$x = \frac{3}{2}$ or $x = -1$

CPs are $\left(\frac{3}{2}, \frac{9}{4}\right)$ ①, $(-1, 1)$ ①

$f_{xx} = 6x \qquad f_{yy} = 6 \qquad f_{xy} = -3$ ①

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$ ① $= 36x - 9$

CP	D	f_{xx}	
$\left(\frac{3}{2}, \frac{9}{4}\right)$	$54 - 9 > 0$ ①	$9 > 0$	f has a relative minimum at $\left(\frac{3}{2}, \frac{9}{4}\right)$ ①
$(-1, 1)$	$-45 < 0$ ①		no relative extremum at ~~$\left(\frac{3}{2}, \frac{9}{4}\right)$~~ $(-1,1)$ ①

(The relative minimum is $f\left(\frac{3}{2}, \frac{9}{4}\right) = \frac{27}{8} - \frac{81}{8} + \frac{243}{16} - \frac{81}{4}$

$= -\frac{189}{16}$

(optional) $= -11.8125$)

2 f_x, f_y

2 CPs

2 $D = 36x - 9$

2 min

2 saddle

Antiderivatives

$\int f(u)\ \mathrm{d}u$ denotes the general antiderivative of $f(u)$.

If $\int f(u)\ \mathrm{d}u = F(u) + c$ then $\dfrac{\mathrm{d}F(u)}{\mathrm{d}u} = f(u)$.

$$\int u^n\ \mathrm{d}u = \frac{u^{n+1}}{n+1} + c,\ n \neq -1$$

$$\int \frac{\mathrm{d}u}{u} = \ln|u| + c$$

$$\int \mathrm{e}^u\ \mathrm{d}u = \mathrm{e}^u + c$$

$$\int a^u\ \mathrm{d}u = \frac{a^u}{\ln a} + c,\ 1 \neq a > 0$$

$$\int \sin u\ \mathrm{d}u = -\cos u + c$$

$$\int \cos u\ \mathrm{d}u = \sin u + c$$

$$\int \sec^2 u\ \mathrm{d}u = \tan u + c$$

$$\int \csc^2 u\ \mathrm{d}u = -\cot u + c$$

$$\int \sec u \tan u\ \mathrm{d}u = \sec u + c$$

$$\int \csc u \cot u\ \mathrm{d}u = -\csc u + c$$

$$\int \tan u\ \mathrm{d}u = \ln|\sec u| + c$$

$$\int \cot u\ \mathrm{d}u = \ln|\sin u| + c$$

$$\int \sec u\ \mathrm{d}u = \ln|\sec u + \tan u| + c$$

$$\int \csc u\ \mathrm{d}u = -\ln|\csc u + \cot u| + c$$

$$\int \sinh u\ \mathrm{d}u = \cosh u + c$$

$$\int \cosh u\ \mathrm{d}u = \sinh u + c$$

$$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c,\ a > 0$$

$$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c,\ a > 0$$

$$\int \frac{\mathrm{d}u}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

$$\int \frac{\mathrm{d}u}{u^2+a^2} = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

$$\sin(x+y) = \sin x\ \cos y + \cos x\ \sin y$$

$$\cos(x+y) = \cos x\ \cos y - \sin x\ \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\ \tan y}$$

$$\sin 2x = 2\sin x\ \cos x$$

$$\cos 2x = 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$