

Name: SOLUTIONS

WILFRID LAURIER UNIVERSITY

Waterloo, Ontario

Mathematics 103 – Calculus I

Midterm – February 26, 2014

Instructor: *Dr. Yuming Chen*

Time Allowed: *80 minutes*

Total Value: *75 marks*

Number of Pages: *6 plus cover page*

Instructions:

Non-programmable, non-graphing calculators are permitted. No other aids are allowed.

Check that your test paper has no missing, blank, or illegible pages.

*Answer in the spaces provided. **Please note that questions are printed on both sides of the test pages.***

Show all your work. Insufficient justification will result in a loss of marks.

- [5 marks] 1. Given $f(x) = x^2$ and $g(x) = \sqrt{x+1}$, calculate $f \circ g$ and specify its domain of definition.

Solution By definition,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (\sqrt{x+1})^2 = x+1.$$

Note the domains of definition for f and g are \mathbb{R} and $[-1, \infty)$, respectively. Hence the domain of definition for $f \circ g$ consists of all $x \in [-1, \infty)$ such that $x^2 \in \mathbb{R}$. It follows that the domain of definition for $f \circ g$ is $[-1, \infty)$.

- [6 marks] 2. Show that $f(x) = x^2 - x$, $x \leq \frac{1}{2}$ is one-to-one, and find a formula for its inverse $f^{-1}(x)$.

Solution Suppose that $x_1, x_2 \leq \frac{1}{2}$ such that $f(x_1) = f(x_2)$. Then

$$x_1^2 - x_1 = x_2^2 - x_2.$$

It follows that

$$\begin{aligned} 0 &= (x_1^2 - x_1) - (x_2^2 - x_2) \\ &= (x_1^2 - x_2^2) - (x_1 - x_2) \\ &= (x_1 - x_2)(x_1 + x_2) - (x_1 - x_2) \\ &= (x_1 - x_2)(x_1 + x_2 - 1). \end{aligned}$$

So $x_1 - x_2 = 0$ or $x_1 + x_2 - 1 = 0$. Since $x_1 \leq \frac{1}{2}$ and $x_2 \leq \frac{1}{2}$, we must have $x_1 = x_2$. This proves that f is one-to-one. The inverse $f^{-1}(x)$ is found by interchanging x and y and solving y as follows:

$$x = y^2 - y$$

or

$$y^2 - y - x = 0$$

gives $y = \frac{1 \pm \sqrt{1+4x}}{2}$. Noting $y \leq \frac{1}{2}$, we have the inverse

$$f^{-1}(x) = \frac{1 - \sqrt{1+4x}}{2}.$$

- [7 marks] 3. Is there a number c such that $\lim_{x \rightarrow -2} \frac{x^2 + cx + c + 1}{x^2 + x - 2}$ exists? If so, find the value of c and the value of the limit.

Solution Note that $f(x) = \frac{x^2 + cx + c + 1}{x^2 + x - 2}$ is a rational function with $(-2)^2 + (-2) - 2 = 0$. So if $\lim_{x \rightarrow -2} f(x)$ exists then we must have

$$(-2)^2 + c(-2) + c + 1 = 0$$

or $c = 5$. In this case,

$$\begin{aligned} \lim_{x \rightarrow -2} f(x) &= \lim_{x \rightarrow -2} \frac{x^2 + 5x + 5 + 1}{x^2 + x - 2} \\ &= \lim_{x \rightarrow -2} \frac{(x + 2)(x + 3)}{(x + 2)(x - 1)} \\ &= \lim_{x \rightarrow -2} \frac{x + 3}{x - 1} \\ &= \frac{-2 + 3}{-2 - 1} \\ &= -\frac{1}{3}. \end{aligned}$$

- [6 marks] 4. Find m so that $g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \geq 3 \end{cases}$ is continuous for all x .

Solution Note that $g(x)$ is a piecewise-defined function and each piece is continuous. For $g(x)$ to be continuous for all x , it suffices that g is continuous at $x = 3$. Since

$$\lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3^-} (x - m) = 3 - m$$

and

$$\lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^+} (1 - mx) = 1 - 3m = g(3),$$

we must have $3 - m = 1 - 3m$ or $m = -1$.

Handwritten notes:
 $4m + 0$ for
~~limit~~
 of the solid line
 are upper part
 (1)

5. Evaluate each of the following limits:

[5 marks]

(a) $\lim_{x \rightarrow 3} \frac{\sqrt{x+6}-3}{x-3}$.

Solution With rationalization, we get

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sqrt{x+6}-3}{x-3} &= \lim_{x \rightarrow 3} \frac{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}{(x-3)(\sqrt{x+6}+3)} \dots (2) \\ &= \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(\sqrt{x+6}+3)} \\ &= \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+6}+3} \dots (1) \\ &= \frac{1}{\sqrt{3+6}+3} \dots (1) \\ &= \frac{1}{6}. \dots (1) \end{aligned}$$

[5 marks]

(b) $\lim_{\theta \rightarrow 0} \frac{\theta^2}{1-\cos \theta}$.

[Hint: $\cos(2\alpha) = 1 - 2\sin^2 \alpha$ for $\alpha \in \mathbb{R}$.]

Solution The limit is evaluated as follows.

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\theta^2}{1-\cos \theta} &= \lim_{\theta \rightarrow 0} \frac{\theta^2}{1-(1-2\sin^2 \frac{\theta}{2})} \dots (2) \\ &= \lim_{\theta \rightarrow 0} \frac{\theta^2}{2\sin^2 \frac{\theta}{2}} \\ &= \lim_{\theta \rightarrow 0} 2 \cdot \left(\frac{\frac{\theta}{2}}{\sin \frac{\theta}{2}}\right)^2 \dots (2) \\ &= 2 \cdot 1^2 \\ &= 2. \dots (1) \end{aligned}$$

[6 marks]

6. Use the **Intermediate Value Theorem** to show that the equation $x^5 - x^3 + 3x - 5 = 0$ has a root in the interval $(1, 2)$.

Solution Let $f(x) = x^5 - x^3 + 3x - 5$. Then f is continuous on the closed interval $[1, 2]$. Note that

(2) $f(1) = 1^5 - 1^3 + 3 \cdot 1 - 5 = -2 < 0$... (1)

and

$f(2) = 2^5 - 2^3 + 3 \cdot 2 - 5 = 25 > 0$ (1)

By the Intermediate Value Theorem, there exists $x_0 \in (1, 2)$ such that $f(x_0) = 0$, that is, $x^5 - x^3 + 3x - 5 = 0$ has a root in the interval $(1, 2)$ (2)

5 marks] 7. Use the definition of a derivative to find $f'(2)$, where $f(x) = x^2 - 2x$.

Solution By definition,

$$\begin{aligned}
 f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 2 \cdot (2+h)] - (2^2 - 2 \cdot 2)}{h} \dots \textcircled{2} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 4 - 2h}{h} \\
 &= \lim_{h \rightarrow 0} (2+h) \dots \textcircled{2} \\
 &= 2. \dots \textcircled{1}
 \end{aligned}$$

[7 marks] 8. Determine an equation of the line tangent to $f(x) = \frac{xe^x}{x+1}$ at $x = 1$.

Solution First,

$$\begin{aligned}
 f'(x) &= \frac{(x+1) \frac{d}{dx}(xe^x) - xe^x \frac{d}{dx}(x+1)}{(x+1)^2} \dots \textcircled{2} \text{ for the quotient rule} \\
 &= \frac{(x+1)(e^x + xe^x) - xe^x}{(x+1)^2} \dots \textcircled{1} \text{ for the product rule} \\
 &\dots \textcircled{1} \text{ for the derivative of } e^x \\
 &\dots \textcircled{1} \text{ for the derivatives of } x \text{ and } x+1
 \end{aligned}$$

Then the slope of the tangent line at $x = 1$ is

$$f'(1) = \frac{(1+1)(e^1 + 1 \cdot e^1) - 1 \cdot e^1}{(1+1)^2} = \frac{3}{4}e. \dots \textcircled{1}$$

As $f(1) = \frac{1}{2}e$, an equation of the tangent line is

$$y - \frac{1}{2}e = \frac{3}{4}e(x - 1) \dots \textcircled{1}$$

or

$$y = \frac{3}{4}ex - \frac{1}{4}e.$$

[5 marks] 9. Let $y = f(x) = x^2 + 2 \sin(x)$. Find $y''(0)$.

Solution First,

$$y'(x) = 2x + 2 \cos x.$$

Then

$$y''(x) = 2 - 2 \sin x.$$

Therefore,

$$y''(0) = 2 - 2 \sin 0 = 2.$$

[7 marks] 10. Use logarithmic differentiation to find the derivative of $y = (x^2 + 1)^{\tan x}$.

Solution Take ln to get

$$\ln y = \ln(x^2 + 1)^{\tan x} = \tan x \ln(x^2 + 1).$$

Differentiating with respect to x gives

$$\frac{y'}{y} = \sec^2 x \ln(x^2 + 1) + \frac{2x \tan x}{x^2 + 1}$$

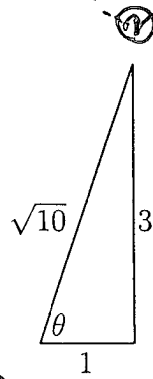
It follows that

$$y' = (x^2 + 1)^{\tan x} \left[\sec^2 x \ln(x^2 + 1) + \frac{2x \tan x}{x^2 + 1} \right]$$

Handwritten notes:
 - (1) for product rule
 - (1) for the derivative of $\ln x$
 - (1) for the chain rule

5 marks] 11. Suppose that $x \in (\pi, \frac{3}{2}\pi)$ and $\tan(x) = 3$. Find the value of $\cos(x)$.

Solution Let $\theta \in [0, \frac{\pi}{2}]$ such that $\tan \theta = 3$. From the reference right triangle below,



we see that $\cos \theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$. Since $x \in (\pi, \frac{3}{2}\pi)$, we have

$$\cos x = -\cos \theta = -\frac{\sqrt{10}}{10} \quad \text{--- (1)}$$

[6 marks] 12. Find y' if $\arctan(x^2y) = y$.

Solution Differentiating implicitly produces

$$\frac{1}{1 + (x^2y)^2} (2xy + x^2y') = y' \quad \text{--- (1)}$$

or

$$2xy + x^2y' = y'(1 + x^4y^2).$$

It follows that

$$y' = \frac{2xy}{1 + x^4y^2 - x^2} \quad \text{--- (1)}$$

for the derivative of arctan and the chain rule