[*5 marks*] 1. Give the precise $\epsilon - \delta$ definition of the statement that the function f has limit L at $x = a$.

[*8 marks*] 2. Find a value $\delta > 0$ such that for all $0 <| x - 1 |< \delta$, $| \ln x - \ln 1 |=| \ln x |< 1/10$. Hint: Consider the 2 cases, $x > 1$ and $x < 1$. If $| \ln x |< 1/10$, find appropriate limits on x in each case, then take the minimum.

[*13 marks*] 3. Evaluate each of the following derivatives:

(a) $\frac{d}{dx}\ \tan(\sin(x))$

(b) $\frac{d}{dt}\ \int_1^t \frac{1}{x^3+x+1}\, dx$

(c) $\frac{d}{dy} \int_{\cos^2 y}^{\sin^2 y} \frac{1}{t}\, dt$

(d) $\frac{d}{dx} \int_1^3 \cos(e^y)\, dy$

(e) $\frac{dy}{dx}$, where $x^2 + \sin(xy) + y^3 = 0$

[*10 marks*] 4. Evaluate the integral $\int_1^2 (x^2+1)dx$ as the limit of Riemann sums with regular partitioning (i.e, equal subdivisions) of $[1, 2]$. Note that $\sum_{i=1}^{n} i = n(n+1)/2$ and $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

[*13 marks*] 5. Evaluate each of the following integrals:

(a) $\int_0^2 (2t^2 + 2t - 2)\, dt$

(b) $\int_1^3 e^{2x}\, dx$

(c) $\int_0^1 \tan^3(x)\, dx + \int_1^0 \tan^3(x)\, dx$

(d) $\int_7^7 \sin^{-1}(x/8)\, dx$

(e) $\int_{-2}^2 x\sqrt{1 - x^2}\, dx$

(f) $\int_{\pi/6}^{\pi/4} \tan^2(x)\, dx$

[*14 marks*] 6. Let $y = xe^{-x}$, $x \geq 0$. For this function, determine:

(a) All horizontal and vertical asymptotes.

(b) All intercepts.

(c) The derivative of y with respect to x.

(d) The critical numbers of y, and, for each critical number, whether it corresponds to a relative maximum, relative minimum or neither.

(e) The absolute maximum and absolute minimum of y, if they exist.

(f) The intervals of monotonicity, that is, the intervals of increase and decrease.

(g) The second derivative of y with respect to x.

(h) Determine the interval(s) where y is concave up, the interval(s) where y is concave down, and any points of inflection.

[*5 marks*] 7. Evaluate the integral :

$$\int_0^{1/2} \sin^{-1}(x)\,dx.$$

[*8 marks*] 8. Find the area of the region bounded by the curves $y = \cos x$, $y = \sin x$, $x = 0$ and $x = \pi/2$.

[*5 marks*] 9. (a) Let P=pressure, V=volume, m=fixed (constant) mass, T=temperature and R=a constant. In an ideal gas, PV=mRT. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1.$$

[*6 marks*] 10. (b) Let $z = \sin(xy)$. Determine:

$$\frac{\partial^2 z}{\partial y \partial x}$$

[*13 marks*] 11. Let $z = 2xy - x^2 - \frac{1}{2}y^4$. Determine all critical points of z, and use the Second Derivative Test to determine whether each critical point corresponds to a relative maximum, relative minimum or a saddle point for z.

Antiderivatives

$\int f(u)\,\mathrm{d}u$ denotes the general antiderivative of $f(u)$.

If $\int f(u)\,\mathrm{d}u = F(u) + c$ then $\dfrac{\mathrm{d}F(u)}{\mathrm{d}u} = f(u)$.

$$\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + c,\ n \neq -1$$

$$\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + c$$

$$\int e^u\,\mathrm{d}u = e^u + c$$

$$\int a^u\,\mathrm{d}u = \frac{a^u}{\ln a} + c,\ 1 \neq a > 0$$

$$\int \sin u\,\mathrm{d}u = -\cos u + c$$

$$\int \cos u\,\mathrm{d}u = \sin u + c$$

$$\int \sec^2 u\,\mathrm{d}u = \tan u + c$$

$$\int \csc^2 u\,\mathrm{d}u = -\cot u + c$$

$$\int \sec u \tan u\,\mathrm{d}u = \sec u + c$$

$$\int \csc u \cot u\,\mathrm{d}u = -\csc u + c$$

$$\int \tan u\,\mathrm{d}u = \ln|\sec u| + c$$

$$\int \cot u\,\mathrm{d}u = \ln|\sin u| + c$$

$$\int \sec u\,\mathrm{d}u = \ln|\sec u + \tan u| + c$$

$$\int \csc u\,\mathrm{d}u = -\ln|\csc u + \cot u| + c$$

$$\int \sinh u\,\mathrm{d}u = \cosh u + c$$

$$\int \cosh u\,\mathrm{d}u = \sinh u + c$$

$$\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c,\ a > 0$$

$$\int \frac{\mathrm{d}u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c,\ a > 0$$

$$\int \frac{\mathrm{d}u}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

$$\int \frac{\mathrm{d}u}{u^2 + a^2} = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

$$\csc^2 x - \cot^2 x = 1$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1 + x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$$

Answers to the Final Exam of Fall 2011

1.

2. $\delta = 1 - e^{-1/10}$

3. (a) $\sec^2(\sin(x))\cos x$
   (b) $\frac{1}{t^3+t+1}$
   (c) $2\sec y \csc y$
   (d) 0
   (e) $-\frac{2x+y\cos(xy)}{x\cos(xy)+3y^2}$

4. $\frac{10}{3}$

5. (a) $\frac{16}{3}$
   (b) $\frac{1}{2}(e^6 - e^2)$
   (c) 0
   (d) 0
   (e) 0
   (f) $1 - \frac{\sqrt{3}}{3} - \frac{1}{12}\pi$

6. (a) The horizontal asymptote is $y = 0$ but there is no vertical asymptote.
   (b) Both x-intercept and y-intercept are 0.
   (c) $y' = (1 - x)e^{-x}$
   (d) The only critical value of y is $x = 1$ and $y(1) = 1/e$ is a relative maximum
   (e) The absolute minimum is 0 and the absolute maximum is $1/e$
   (f) y is increasing on $(0, 1)$ and is decreasing on $(1, \infty)$
   (g) $y'' = (x - 2)e^{-x}$
   (h) y is concave down on $(0, 2)$ and is concave up on $(2, \infty)$. $(2, y(2)) = (2, \frac{2}{e^2})$ is the point of inflection

7. $\frac{1}{12}\pi + \frac{\sqrt{3}}{2} - 1$

8. $2\sqrt{2} - 2$

9. (a)

10. (b) $\cos(xy) - xy\sin(xy)$

11. (not covered)