CONCORDIA UNIVERSITY
Department of Mathematics and Statistics

Course	**Number**	**All sections**
MATH	251/2	
Practice final 2	**SOLUTIONS**	

1. (a)

$$(\mathbf{u}, A\mathbf{v}) = \sum_{i=1}^{n}\sum_{jh=1}^{n} u_i A_{ij} v_j, \quad (A\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ji} u_i v_j$$

therefore $(\mathbf{u}, A\mathbf{v}) = (A\mathbf{u}, \mathbf{v})$ is true for all $\mathbf{u}, \mathbf{v}$ if and only if $A = A^T$. (The "if" is immediate from the equation, the "only if" follows by applying this equality to all pairs of basis vectors.)

(b) If $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors with eigenvalues λ_1, λ_2), we have

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2.$$

Taking the scalar products, and using the symmetry of A, we have

$$(\mathbf{v}_1, A\mathbf{v}_2) = \lambda_2(\mathbf{v}_1, \mathbf{v}_2) = (A\mathbf{v}_1, \mathbf{v}_2) = \lambda_1(\mathbf{v}_1, \mathbf{v}_2).$$

Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{v}_1, \mathbf{v}_2) = 0$$

Since $\lambda_1 \neq \lambda_2$, we have

$$(\mathbf{v}_1, \mathbf{v}_2) = 0.$$

Now, take the complex conjugate of the eigenvector equation

$$A\mathbf{v} = \lambda \mathbf{v}, \quad A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}},$$

So $\overline{\mathbf{v}}$ is also an eigenvector, with eigenvalue $\overline{\lambda}$. If $\lambda \neq \overline{\lambda}$, we have, by the above,

$$(\mathbf{v}, \overline{\mathbf{v}}) = \sum_{i=1}^{n} |v_i|^2 = 0, \to \quad v_i = 0, \quad i = 1, \ldots, n.$$

and hence $\mathbf{v} = \mathbf{0}$ is the zero vector, not an eigenvector. Therefore $\lambda = \overline{\lambda}$ is real.

(c) Suppose $\mathbf{v}$ is an eigenvector and $\mathbf{u} \in U_{\mathbf{v}}$, then

$$(\mathbf{u}, \mathbf{v}) = 0$$

By the symmetry of $A = A^T$, we have

$$(A\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) = 0,$$

and hence $A\mathbf{u} \in U_{\mathbf{v}}$. Therefore, all vectors in $U_{\mathbf{v}}$ are orthogonal to $\mathbf{v}$. This implies $U_{\mathbf{v}}$ is orthogonal and complementary to $\text{span}\{\mathbf{v}\}$.

$$\mathbf{R}^n = U_{\mathbf{v}} \oplus \text{span}\{\mathbf{v}\}.$$

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2. (a) The map is linear: $\Phi(M + \lambda N) = B(M + \lambda N)B^{-1} = BMB^{-1} + \lambda BNB^{-1} = \Phi(M) + \lambda\Phi(N)$, where we have used that the matrix multiplication is distributive.

Since B is invertible, we have $BMB^{-1} = Q$ if and only if $M = B^{-1}QB$. So, the inverse is $\Phi^{-1}(Q) = B^{-1}QB$.

(b) Recalling that

$$B^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{0.1}$$

we find

$$B^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \tag{0.2}$$

$$\Phi(E_{11}) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 2 & -4 \end{bmatrix} \tag{0.3}$$

$$\Phi(E_{12}) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -10 & 25 \\ -4 & 10 \end{bmatrix} \tag{0.4}$$

$$\Phi(E_{21}) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \tag{0.5}$$

$$\Phi(E_{22}) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix} \tag{0.6}$$

Therefore the matrix representation in the standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is

$$[\Phi] = \begin{bmatrix} 5 & -10 & 2 & -4 \\ -10 & 25 & -4 & 10 \\ 2 & -4 & 1 & -2 \\ -4 & 10 & -2 & 5 \end{bmatrix} \tag{0.7}$$

[10] **3.** Define the 4×5 matrix A that has the elements of S as its columns

$$A = \begin{pmatrix} -1 & 5 & -2 & 4 & 3 \\ -3 & 15 & -1 & 16 & 8 \\ 4 & -20 & 2 & -22 & -12 \\ 2 & -10 & -3 & -7 & 2 \end{pmatrix}.$$

Its row reduced echelon form is then

$$\begin{pmatrix} 1 & -5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, S is not linearly independent, but a maximal linearly independent subset is given by the first, third and fourth columns:

$$S' = \left\{ \begin{pmatrix} -1 \\ -3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ -22 \\ -7 \end{pmatrix} \right\}.$$

Since the first, third and fourth columns of the row reduced echelon form of the matrix are e_1, e_2, e_3, we conclude that adding e_4 completes these columns of A to a basis; i.e.

$$S' = \left\{ \begin{pmatrix} -1 \\ -3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ -22 \\ -7 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbf{R}^4$.

4a. The solution set is $S = \{(1 - x_2 + \frac{1}{2}x_4,\ x_2,\ 1 + \frac{1}{2}x_4,\ x_4)\} = \{(1,\ 0,\ 1,\ 0) + t(-1,\ 1,\ 0,\ 0) + s(0,\ 0,\ 1,\ 2),\ t, s \in \mathbf{R}\}$.

b. By (a), a basis for the solution set of the corresponding homogeneous system is $\{(-1,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 2)\}$.

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5 a.

$$\det(A - \lambda \mathbf{I}) = -\lambda^3 + 6\lambda^2 + 35\lambda + 38.$$

b.

$$\det(B - \lambda \mathbf{I}) = -\lambda^3 - \lambda^2 - 8\lambda - 62.$$

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6. Solving the characteristic equation

$$\det \begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 4 = 0,$$

the roots are the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$. Solving the corresponding eigenvector equations

$$2x + 2y = x, x, \quad x + 3y = y$$

and

$$2x + 2y = 4x, x, \quad x + 3y = 4y$$

gives the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

Therefore a choice for the nonsingular matrix Q is

$$Q = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix},$$

giving

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

But any other matrix of the form

$$Q = \begin{pmatrix} 2a & 2b \\ -a & 4b \end{pmatrix},$$

for (a, b) any nonzero constants, will do just as well.

[3] **Bonus question.** Let $\mathbf{v}_1$ be an eigenvecxtor of A. Since $U_{\mathbf{v}_1}$ consists of vectors orthogonal to $\mathbf{v}_1$ and is invariant under A, we have

$$AU_{\mathbf{v}_1} \subset U_{\mathbf{v}_1}$$

But the restriction of A to the subspace $U_{\mathbf{v}_1}$ is also symmetric, since

$$(A\mathbf{u}, \mathbf{v}) = (\mathbf{u}, A\mathbf{v}) \quad \forall\ \mathbf{u}, \mathbf{v} \in U_{\mathbf{v}_1}.$$

Therefore there exists another eigenvector $\mathbf{v}_2 \in U_{\mathbf{v}_1}$ that is orthogonal to $\mathbf{v}_1$. Continuing in this way, by considering smaller and smaller subspaces that are orthogonal to the consecutive eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ we obtain an entire basis of (orthogonal) eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We can choose to normalize all these to length 1, by dividing by their lengths. The matrix Q is then the one whose column vectors are the $\mathbf{v}_i$'s.

$$Q \begin{pmatrix} \mathbf{v}_1, & \cdots & \mathbf{v}_n \end{pmatrix}.$$

The orthogonality condition

$$Q^T Q = Q Q^t = \mathbf{I}_n$$

is just a matrix way to summarize the orthogonality relations

$$(\mathbf{v}_i, \mathbf{v}_j) = 0, \quad i, j = 1, j = 1, \ldots, n, \quad \text{if} \quad i \neq j, \quad (\mathbf{v}_i, \mathbf{v}_i) = 1.$$