CONCORDIA UNIVERSITY
Department of Mathematics and Statistics

Course	**Number**	**All sections**
MATH	251/2	
Practice final 1	**SOLUTIONS**	

1.

a. Taking a linear combination $ap_1 + bp_2$ of two polynomials (p_1, p_2), we have

$$\begin{aligned} T(ap_1+bp_2)(x) &= xap_1(1) + xbp_2(1) - ap_1(0) - bp_2(0) - \int_1^x ap_1(y)dy - \int_1^x bp_2(y)dy \\ &= axp_1(1) - ap_1(0) - a\int_1^x p_1(y)dy + bxp_2(1) - bp_2(0) - b\int_1^x p_2(y))dy \\ &= aT(p_1(x)) + bT(p_2(x)). \end{aligned}$$

b.

$$\begin{gathered} T(1) = x - 1 - \int_1^x dy = 0, \quad T(x) = x - \int_1^x ydy = x + \frac{1-x^2}{2}, \\ T(x^2) = x - \int_1^x y^2dy = x + \frac{1-x^3}{3}, \quad T(x^3) = x - \int_1^x y^3dy = x + \frac{1-x^4}{4}. \end{gathered}$$

Therefore $N(T) = \text{span}\{1\}$.

c.

$$R(T) = \text{span}\{x^2 - 2x - 1, x^3 - 3x - 1, x^4 - 4x - 1\}.$$

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2. Let v_1 be a vector spanning the line L and v_2 a perpendicular vector; from elementary geometry, a possible choice is

$$v_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}, \qquad v_2 = \begin{bmatrix} m \\ -1 \end{bmatrix} \tag{0.1}$$

The matrix of change of coordinates from the standard basis σ to the basis $\beta = \{v_1, v_2\}$ is

$$[Id]_\beta^\sigma = Q = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}, \qquad [Id]_\sigma^\beta = Q^{-1} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} \tag{0.2}$$

The answers to the two questions in the new basis are

$$(\mathbf{1})\ [T]_\beta = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (\mathbf{2})\ [T]_\beta = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{0.3}$$

Consequently, back in the standard basis, the answers are:

$$(\mathbf{1})[T]_\sigma = [Id]^\sigma_\beta [T]_\beta [Id]^\beta_\sigma = \frac{1}{1+m^2}\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix} \tag{0.4}$$

$$(\mathbf{2})[T]_\sigma = [Id]^\sigma_\beta [T]_\beta [Id]^\beta_\sigma = \frac{1}{1+m^2}\begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+m^2} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{bmatrix} \tag{0.5}$$

3a. We have

$$rank(\lambda A) = rank(L_{\lambda A}) = rank(\lambda L_A) = dim R(\lambda L_A) = dim R(L_A) = rank(L_A) = rank(A),$$

where $L_A : F^n \to F^m$ is the left-multiplication linear transformation which satisfies: $L_{\lambda A} = \lambda L_A$.

3b. The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so, $rank(A) = 3$, which is the maximum number of its linearly independent columns; that is, $rank(A) = 3$ is the dimension of the subspace generated by its columns. Since the first, second and third columns of rref(A) are e_1, e_2 and e_3, we have that the first, second and third columns of A are linearly independent.

4a. $det(A) = (b-a)(c-a)(c-b)$, which is nonzero since a, b and c are distinct. Thus, A is invertible and

$$A^{-1} = \frac{\tilde{A}}{\det(A)}$$

where $\tilde{A}$ is the classical adjoint of A. Therefore,

$$A^{-1} = \frac{1}{(b-a)(c-a)(c-b)}\begin{pmatrix} bc(c-b) & b^2-c^2 & c-b \\ ac(a-c) & c^2-a^2 & a-c \\ ab(b-a) & a^2-b^2 & b-a \end{pmatrix}.$$

4b. The coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix},$$

with $det(A) \neq 0$ (by a), and therefore, the system has a unique solution given by:

$$\begin{aligned}
x &= \frac{1}{(b-a)(c-a)(c-b)} \det \begin{pmatrix} 1 & 1 & 1 \\ m & b & c \\ m^2 & b^2 & c^2 \end{pmatrix} = \frac{(b-m)(c-m)}{(b-a)(c-a)} \\
y &= \frac{1}{(b-a)(c-a)(c-b)} \det \begin{pmatrix} 1 & 1 & 1 \\ a & m & c \\ a^2 & m^2 & c^2 \end{pmatrix} = \frac{(m-a)(c-m)}{(b-a)(c-b)} \\
z &= \frac{1}{(b-a)(c-a)(c-b)} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & m \\ a^2 & b^2 & m^2 \end{pmatrix} = \frac{(m-a)(m-b)}{(c-a)(c-b)}.
\end{aligned}$$

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5 a. The characteristic equation is

$$\det(A - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 2)^2 = 0.$$

The roots are therefore $\lambda_1 = -1$ (single) and $\lambda_2 = 2$ (double).

b. Solving the corresponding eigenvector equations for $\lambda_1 = -1$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = - \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives the 1-d space of eigenvectors span$\{\mathbf{v}_1\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

while solving them for $\lambda_2 = 2$,

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives the 2-d space of eigenvectors span$\{\mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

We therefore have the basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

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6. Diagonalizing the matrix of coefficients

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

we obtain

$$Q^{-1}AQ = D,$$

where the diagonal matrix of eigenvalues is

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+2\sqrt{2} & 0 \\ 0 & 0 & 1-2\sqrt{2} \end{pmatrix}$$

while

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}$$

is the matrix whose columns are linearly independent eigenvectors of A. Defining the column vector $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Q \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

we get

$$\frac{dX}{dt} = X, \quad \frac{dY}{dt} = (1+2\sqrt{2})Y, \quad \frac{dZ}{dt} = (1-2\sqrt{2})Z,$$

which integrate to

$$X = ae^t, \quad Y = be^{(1+2\sqrt{2})t} \quad Z = ce^{(1-2\sqrt{2})t}$$

for arbitrary constants (a, b, c). Therefore the general solution is

$$\begin{aligned} x(t) &= ae^t + be^{(1+2\sqrt{2})t} + ce^{(1-2\sqrt{2})t} \\ y(t) &= \sqrt{2}be^{(1+2\sqrt{2})t} - \sqrt{2}ce^{(1-2\sqrt{2})t} \\ z(t) &= -ae^t + be^{(1+2\sqrt{2})t} + ce^{(1-2\sqrt{2})t}. \end{aligned}$$