**CONCORDIA UNIVERSITY**
**Department of Mathematics and Statistics**

| **Course** | **Number** | | **All sections** |
|---|---|---|---|
| MATH | 251/2 | | |
| **Examination** | **Date** | **Time** | **Pages** |
| Practice final 2 | December 10, 2016 | 14:00 - 17:00 | 3 |

**Course examiner:** Prof. J. Harnad

**Instructions:** Answer all parts of all numbered questions 1-6. The value for each part is indicated in square brackets in the margin (out of a possible total of 60). Lined examination booklets will be provided. Write all relevant calculations, proofs and results on the right hand pages. Left hand pages are for rough work only, and will not be read or included in the grading. Only calculators of the type authorized by the Department of Mathematics and Statistics may be used. Any books, notes, or other recorded materials may not be used.

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**1.** The scalar product of any two (column) vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ is defined as

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^{n} u_i v_i.$$

where $\{u_i\}_{i=1,\dots,n}$, $\{v_i\}_{i=1,\dots,n}$ are the components of $\mathbf{u}$ and $\mathbf{v}$ in the standard basis. The length $|\mathbf{v}|$ of a vector $\mathbf{v}$ is

$$|\mathbf{v}| = \sqrt{(\mathbf{v}, \mathbf{v})}.$$

Two vectors are *orthogonal* (or *perpendicular*) if their scalar product vanishes.

$$(\mathbf{u}, \mathbf{v}) = 0.$$

A matrix $A \in \mathrm{Mat}^{n\times n}(\mathbf{R})$ is *symmetric* if it equals its transpose

$$A = A^T$$

(or, equivalently $A_{ij} = A_{ji}$ for all $i, j, = 1, \dots, n$).

[2] **a.** Prove that $A$ is symmetric if and only if

$$(\mathbf{u}, A\mathbf{v}) = (A\mathbf{u}, \mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$.

[4] **b.** Prove that any two eigenvectors of a symmetric matrix $A$ with different eigenvalues are orthogonal, and hence, that all eigenvalues are real. (Hint, take the scalar product of one eigenvector $\mathbf{u}$ with $A$ applied to the other on $\mathbf{v}$, and use the above property. Then take the complex conjugate of the eigenvector equation and show the eigenvalue must equal its complex conjugate.)

[4] **c.** Define the subspace $U_{\mathbf{v}} \subset \mathbf{R}^n$ to consist of all vectors orthogonal to $\mathbf{v}$

$$U_{\mathbf{v}} := \{\mathbf{u} \in \mathbf{R}^n, (\mathbf{u}, \mathbf{v}) = 0.$$

Prove that:

1. $\dim(U_{\mathbf{v}}) = n - 1$ and $\mathbf{R}^n$ is the (orthogonal) direct sum $\mathbf{R}^n = U_{\mathbf{v}} \oplus \operatorname{span}\{\mathbf{v}\}$.
2. The spaces $U_{\mathbf{v}}$ and $\operatorname{span}\{\mathbf{v}\}$ are each invariant under $A$.

**2.** Let $B \in Mat_{n\times n}(\mathbb{R})$ be an invertible matrix.

[5] **a.** Show that $\Phi : Mat_{n\times n}(\mathbb{R}) \to Mat_{n\times n}(\mathbb{R})$ defined by $\Phi(M) = BMB^{-1}$ is a linear invertible map and find its inverse.

[5] **b.** Let $B = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Find the matrix representation of $\Phi$ in the standard basis.

[10] **3.** Let

$$S = \left\{ \begin{pmatrix} -1 \\ -3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 15 \\ -20 \\ -10 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ -22 \\ -7 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -12 \\ 2 \end{pmatrix} \right\}.$$

Determine whether $S$ is linearly independent. If not, find a maximal linearly independent subset and extend it to a basis for $V = \mathbf{R}^4$.

**4.** Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 &= 0. \end{aligned}$$

[5] **a.** Determine whether the system is consistent. If the system is consistent, find all solutions.

[5] **b.** Find a basis for the solution set of the corresponding homogeneous system.

**5.** Find the characteristic polynomials of the following matrices

[5] **a.**

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 4 \\ 6 & 4 & 5 \end{pmatrix}$$

[5] **b.**

$$B = \begin{pmatrix} 1 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{pmatrix}.$$

**6.** Let

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

[5] **a.** Find all the eigenvalues and eigenvectors of $A$.

[5] **b.** Find a nonsingular matrix $Q$ such that $Q^{-1}AQ$ is diagonal.

[3] **Bonus question.**

Let $A \in \mathrm{Mat}^{n\times n}(\mathbf{R})$ be a symmetric matrix $A^T = A$, as in question 1. Using the results of question 1, prove that $\mathbf{R}^n$ has a basis of eigenvectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ that are mutually orthogonal

$$(\mathbf{v}_i, \mathbf{v}_j) = 0 \quad \text{if} \quad i \neq j$$

and of unit length: $(\mathbf{v}_i, \mathbf{v}_i) = 1$, and hence, that there exists an orthogonal matrix $Q \in \mathrm{Mat}^{n\times n}(\mathbf{R})$

$$QQ^T = Q^TQ = \mathbf{I}_n$$

such that $Q^TAQ$ is the diagonal matrix of eigenvalues.