1. (1 point) If $f(x) = e^x \arctan(x)$ then $f'(-1) =$

A. 1

B. $-\frac{\pi}{4e}$

C. 0

D. $e^{-1}\left(\frac{1}{2} - \pi\right)$

E. e^{-1}

F. $e^{-1}\left(\frac{1}{2} - \frac{\pi}{4}\right)$

Your answer: F

$f'(x) = e^x \arctan(x) + \frac{e^x}{1+x^2}$

$f'(-1) = e^{-1}\left(-\frac{\pi}{4}\right) + \frac{e^{-1}}{2}$

2. (1 point) Which of the following is equal to the derivative of $f(x) = 2^{\cos(x)}$?

A. $-2^{\cos(x)}\sin(x)$

B. $\ln(2)2^{\sin(x)}\cos(x)$

C. $-2^{\cos(x)}\sin(x)\ln(2)$

D. $\frac{2^{\cos(x)}}{\ln(2)}$

E. $2^{\cos(x)}$

F. $-\cos(x)2^{\cos(x)-1}\sin(x)$

Your answer: C

$\frac{d}{dx} 2^{\cos x} = 2^{\cos(x)} \cdot \ln(2) \cdot (-\sin(x))$

3. (1 point) Which of the following is equal to the derivative of $g(x) = \ln(e^x x^{-3})$?

A. $\frac{x^3}{e^x}$

B. $1 - 3/x$

C. $e^x(x^{-3} - 3x^{-4})$

D. $(x^{-3} - 3x^{-4})e^x \ln(e^x x^{-3})$

E. $e^x \ln(x^{-3} - 3e^x/x)$

F. $-3/x$

Your answer: B

$g(x) = \ln(e^x x^{-3}) = \ln(e^x) + \ln(x^{-3}) = x - 3\ln(x)$

$g'(x) = 1 - \frac{3}{x}$

4. (1 point) Suppose f is a function and we know that

$$\sec(f(x)) + f(x) = x^2$$

for all x in the domain of f. Then

A. $f'(x) = \dfrac{2x}{\sec(x)\tan(x) + 1}$

B. $f'(x) = 2x - \sec(f(x))\tan(f(x))$

C. $f'(x) = \dfrac{2x}{\sec^2(x) + 1}$

D. $f'(x) = \dfrac{2x}{\sec(f(x))\tan(f(x)) + 1}$

E. $f'(x) = 2x - \sec^2(x)$

F. $f'(x) = \dfrac{2x}{\sec^2(f(x)) + 1}$

Your answer: D

$\sec(f(x))\tan(f(x))f'(x) + f'(x) = 2x$

$f'(x) = \dfrac{2x}{\sec(f(x))\tan(f(x)) + 1}$

5. (1 point) Which of the following is the equation of the tangent line to the curve

$$y = x^{2/3} - 32/x$$

at the point $(8, 0)$?

A. $y = \frac{7}{6}x - \frac{28}{3}$

B. $y = \frac{1}{3}x - \frac{8}{3}$

C. $y = \frac{2}{3}x - \frac{16}{3}$

D. $y = \frac{5}{6}x - \frac{20}{3}$

E. $y = x - 8$

F. $y = \frac{2}{3}x^{2/3} + \frac{32}{x} - \frac{20}{3}$

Your answer: D

$y' = \frac{2}{3}x^{-1/3} + 32x^{-2}$

at $x=8$: $y' = \frac{2}{3}\frac{1}{\sqrt[3]{8}} + \frac{32}{8^2} = \frac{2}{3}\left(\frac{1}{2}\right) + \frac{32}{64} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$y = \frac{5}{6}x + b \Rightarrow b = -\frac{5}{6} \cdot 8 = -\frac{20}{3}$

6. (1 point) If $h(x) = \sqrt{e^{2x} + x^4}$ then

A. $h'(x) = \dfrac{2e^{2x} + 4x^3}{2\sqrt{e^{2x} + x^4}}$

B. $h'(x) = \dfrac{2e^{2x} + 4x^3}{2(e^{2x} + x^4)}$

C. $h'(x) = e^x + 2x$

D. $h'(x) = \dfrac{1}{2\sqrt{e^{2x} + x^4}}$

E. $h'(x) = \dfrac{1}{2\sqrt{2e^{2x} + 4x^3}}$

F. $h'(x) = \dfrac{2e^{2x} + 4x^3}{2\sqrt{x}}$

Your answer: A

$h'(x) = \frac{1}{2}(e^{2x} + x^4)^{-1/2}(2e^{2x} + 4x^3)$

7. (1 point) Suppose we are given

$$f(x) = \frac{3 - 6x + x^2}{x}, \quad f'(x) = \frac{x^2 - 3}{x^2}, \quad f''(x) = \frac{6}{x^3}.$$

Which one of the following statements is true of the value

$$x = -\sqrt{3} \ ?$$

A. It is not a critical point of f, but it is an inflection point.

B. f has a local minimum there, but it is not a global extremum.

C. f has a local maximum there, but it is not a global extremum.

D. f has a local minimum and a global maximum there.

E. f has a local and global maximum there.

F. It is neither a critical point nor an inflection point of f.

Your answer: C

$f'(-\sqrt{3}) = 0$ critical point

$f''(-\sqrt{3}) < 0$ local max

$\lim_{x \to \infty} f(x) = \infty$ $\therefore$ not global max

8. (1+3+1=5 points) In this question, you will use the definition of the derivative to find the derivative of a function, and then you will check your answer using the quotient rule.

(a) Give the definition of the derivative of a differentiable function $f(x)$ at a point x in its domain.

Answer: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

(b) Using the definition of the derivative, find the derivative of $f(x) = \dfrac{4x}{3-x}$ at a point x in its domain.

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{4(x+h)}{3-(x+h)} - \frac{4x}{3-x}\right)$$ 0.5

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{4(x+h)(3-x) - 4x(3-x-h)}{(3-x-h)(3-x)}\right)$$ common 0.5

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{4(3x-x^2+3h-hx) - 4(3x-x^2-xh)}{(3-x-h)(3-x)}\right)$$ mult 0.5

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{12h}{(3-x-h)(3-x)}\right)$$ simp 0.5

$$= \lim_{h\to 0} \frac{12}{(3-x-h)(3-x)}$$ cancel 0.5

$$= \frac{12}{(3-x)^2}$$ limit 0.5

(c) Compute the derivative of $f(x) = \dfrac{4x}{3-x}$ using the quotient rule.

$$f'(x) = \frac{(3-x)4 - 4x(-1)}{(3-x)^2} = \frac{12-4x+4x}{(3-x)^2} = \frac{12}{(3-x)^2}$$ ✓

9. (2 + 2 + 1 = 5 points) A population of snails for a nice restaurant grows at a logistic rate, and is harvested regularly. Denoting the fraction of snails harvested each month by the parameter h, the DTDS governing the growth of this population is given by

$$N_{t+1} = N_t(2.4 - N_t) - hN_t,$$

where t is in months, N_t is the population (in thousands of snails) at time t, and $0 \leq h \leq 1$.

(a) Find the equilibria N^* of this system, showing your work below. Your answers may be formulas using the parameter h.

$N^* =$ 0 and $N^* =$ 1.4 − h

$f(N) = N(2.4 - N) - hN$

We solve $f(N) = N$: $N = 2.4N - N^2 - hN$

$\iff N^2 + (1 - 2.4 + h)N = 0$

$N = 0$ or $N = 1.4 - h$

If you were unable to answer (a), use the (incorrect) formula $\frac{4}{17}(4 - 3h)$ for N^* in parts (b) and (c).

(b) The *positive* equilibrium N^* is stable, and the *equilibrium harvest* S is $S = hN^*$. Using your formula in (a), find the value of h that maximizes S, justifying your answer with Calculus.

Optimal rate of harvest h: 0.7

$S = h(1.4 - h)$

$S = 1.4h - h^2$

$S' = 1.4 - 2h$

$S' = 0$ when $2h = 1.4$

$h = 0.7$

So $h = 0.7$ is the only critical point.

Since $S'' = -2$, S is always concave down ∴ the critical point is a local max.

Since S is concave down on $[0, 1]$ the max is a global max.

(c) Find the maximum value of the equilibrium harvest S that the restaurant can expect.

Maximum value of S: 0.49

$S = hN^* = h(1.4 - h)$

$= 0.7(0.7)$

10. (8 points) We would like to sketch the graph of $y = f(x)$. We have computed for you:

$$f(x) = \frac{x}{x^3+1}, \quad f'(x) = \frac{-2x^3+1}{(x^3+1)^2} \quad \text{and} \quad f''(x) = \frac{6x^2(x^3-2)}{(x^3+1)^3}.$$

Fill in the answers below and complete the following tables and then sketch the graph of the function using this information. **Round your values to three decimal places.**

$\lim_{x\to(-1)^+} f(x) =$ $-\infty$ and $\lim_{x\to(-1)^-} f(x) =$ ∞

$\lim_{x\to-\infty} f(x) =$ 0 and $\lim_{x\to\infty} f(x) =$ 0

$-2x^3+1=0$

$\iff 2x^3=1$

$\iff x=\sqrt[3]{\tfrac{1}{2}}$

Critical points of f: $\sqrt[3]{\frac{1}{2}} \sim 0.794$, -1 (vertical asymptote)

Intervals of x	$x<-1$	$-1<x<\sqrt[3]{\frac{1}{2}}$	$x>\sqrt[3]{\frac{1}{2}}$
Sign of $f'(x)$	$+$	$+$	$-$
Behaviour of $y=f(x)$	increasing	increasing	decreasing

$6x^2(x^3-2)=0$

$\iff x=0$ or $x^3=2$

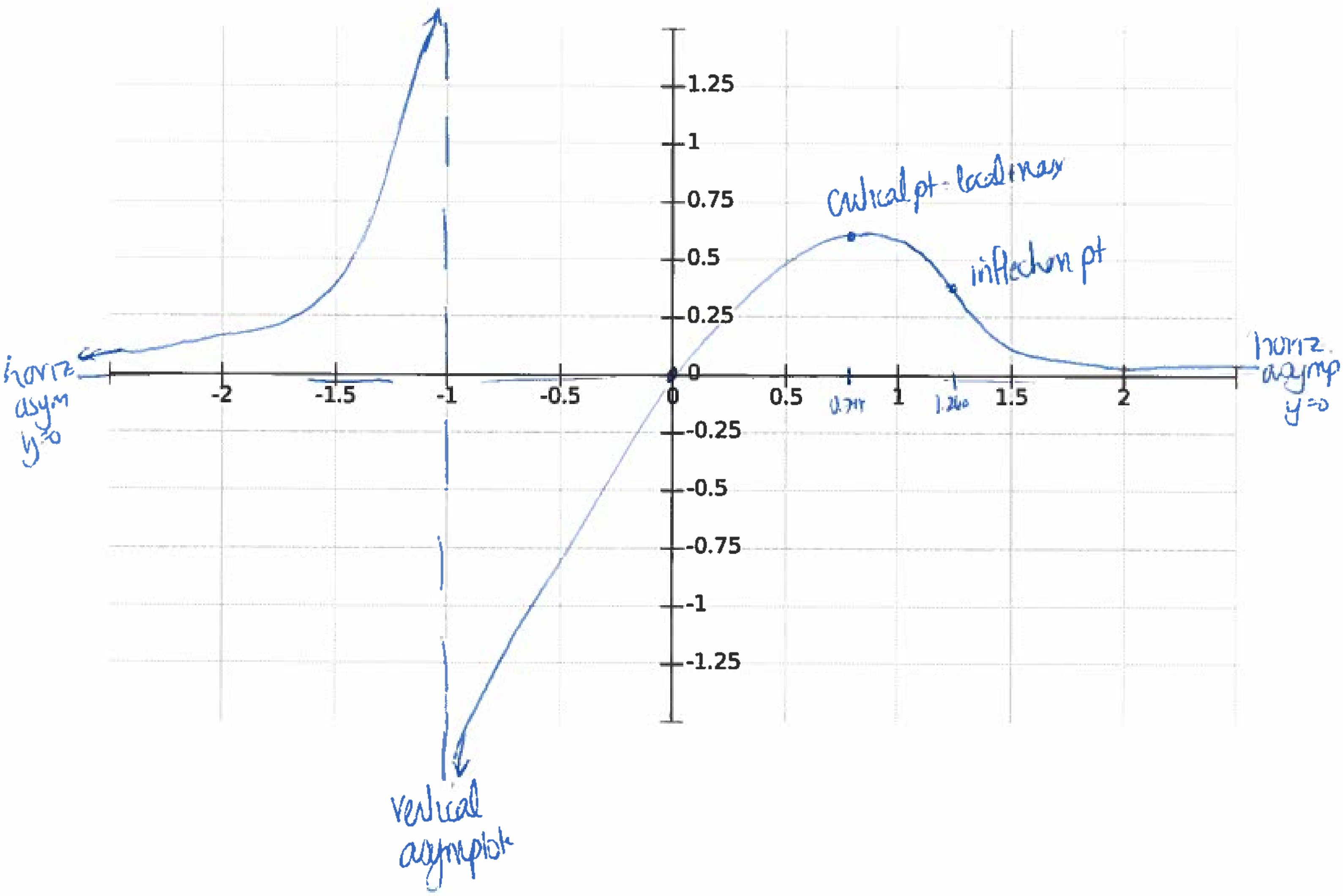
$\iff x=\sqrt[3]{2}$

x-values where concavity of f could change: -1, 0, $\sqrt[3]{2}\sim 1.260$

Intervals of x	$x<-1$	$-1<x<0$	$0<x<\sqrt[3]{2}$	$x>\sqrt[3]{2}$
Sign of $f''(x)$	$+$	$-$	$-$	$+$
Behaviour of $y=f(x)$	concave up	concave down	concave down	concave up

On the following page, sketch a graph of the function $y = f(x)$. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**

On this page, sketch a graph of the function $y = f(x)$ from Question 10. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**



$f(0) = 0$

$f(0.794) \sim 0.529$

$f(1.260) \sim 0.420$

1. (1 point) If $f(x) = e^x \arcsin(x)$ then $f'(\frac{1}{2}) =$

(A.) $e^{1/2}(\pi/6 + 2/\sqrt{3})$
B. $e^{1/2}\pi/6$
C. 0
D. $\frac{2\sqrt{3}e^{1/2}}{3}$
E. 1
F. $e^{1/2}$

Your answer: A

$f'(x) = e^x \arcsin(x) + e^x \frac{1}{\sqrt{1-x^2}}$

$f'(\frac{1}{2}) = e^{1/2}\frac{\pi}{6} + e^{1/2}\frac{1}{\sqrt{1-1/4}} = e^{1/2}\left(\frac{\pi}{6} + \frac{2}{\sqrt{3}}\right)$

$\arcsin(\frac{1}{2}) = \theta$
$\Rightarrow \sin\theta = \frac{1}{2}$
and $-\pi/2 \leq \theta \leq \pi/2$
$\Leftrightarrow \theta = \pi/6$

2. (1 point) Which of the following is equal to the derivative of $f(x) = 3^{\sin(x)}$?

A. $3^{\sin(x)}\cos(x)$
B. $\sin(x)\cos(x)3^{\sin(x)-1}$
C. $3^{\cos(x)}\sin(x)\ln(2)$
D. $\frac{3^{\sin(x)}}{\ln(3)}$
E. $3^{\sin(x)}$
(F.) $\ln(3)3^{\sin(x)}\cos(x)$

Your answer: F

$f'(x) = 3^{\sin(x)}\ln(3)\cos(x)$

3. (1 point) Which of the following is equal to the derivative of $g(x) = \ln(e^x x^4)$?

(A.) $1 + 4/x$
B. $e^x \ln(x^4) - 4e^x/x$
C. $4x^3e^x + x^4e^x$
D. $(4x^3 + x^4)e^x \ln(e^x x^4)$
E. $\frac{e^x}{x^4}$
F. $4/x$

Your answer: A

$g(x) = \ln(e^x x^4) = \ln(e^x) + \ln(x^4) = x + 4\ln(x)$

$g'(x) = 1 + \frac{4}{x}$

4. (1 point) Suppose f is a function and we know that

$$\tan(f(x)) + x^2 = f(x)$$

for all x in the domain of f. Then

A. $f'(x) = \sec(x)\tan(x) - 2x$

B. $f'(x) = \dfrac{2x}{1 - \sec^2(f(x))}$

C. $f'(x) = \dfrac{2x}{1 - \sec(x)\tan(x)}$

D. $f'(x) = \dfrac{2x}{1 - \sec(f(x))\tan(f(x))}$

E. $f'(x) = \sec^2(f(x)) - 2x$

F. $f'(x) = \dfrac{2x}{1 - \sec^2(x)}$

Your answer: B

$\sec^2(f(x))f'(x) + 2x = f'(x)$

$\therefore \quad f'(x)(1 - \sec^2(f(x))) = 2x$

$\therefore \quad f'(x) = \dfrac{2x}{1 - \sec^2(f(x))}$

5. (1 point) Which of the following is the equation of the tangent line to the curve

$$y = x^{1/3} + 16/x$$

at the point $(8, 4)$?

A. $y = -\frac{1}{6}x + \frac{16}{3}$

B. $y = x - 4$

C. $y = \frac{5}{6}x - \frac{8}{3}$

D. $y = \frac{1}{3}x^{1/3} - \frac{16}{x} + \frac{4}{3}$

E. $y = -x + 12$

F. $y = \frac{1}{6}x + \frac{8}{3}$

Your answer: A

$y' = \frac{1}{3}x^{-2/3} - 16x^{-2}$

at $x=8$: $y' = \frac{1}{3}\left(\frac{1}{\sqrt[3]{8}}\right)^2 - \frac{16}{8^2} = \frac{1}{3}\cdot\frac{1}{4} - \frac{16}{64}$

$= \frac{1}{12} - \frac{1}{4} = \frac{1}{12} - \frac{3}{12} = \frac{-1}{6}$

$y = -\frac{1}{6}x + b$ at $(8,4) \Rightarrow 4 = -\frac{8}{6} + b \Rightarrow b = 4 + \frac{4}{3} = \frac{16}{3}$

6. (1 point) If $h(x) = \sqrt{e^{4x} + x^3}$ then

A. $h'(x) = e^{2x} + \frac{3}{2}\sqrt{x}$

B. $h'(x) = \frac{4e^{4x} + 3x^2}{2(e^{4x} + x^3)}$

C. $h'(x) = \frac{1}{2\sqrt{4e^{4x} + 3x^2}}$

D. $h'(x) = \frac{1}{2\sqrt{e^{4x} + x^3}}$

(E.) $h'(x) = \frac{4e^{4x} + 3x^2}{2\sqrt{e^{4x} + x^3}}$

F. $h'(x) = \frac{4e^{4x} + 3x^2}{2\sqrt{x}}$

Your answer: E

$h'(x) = \frac{1}{2}(e^{4x} + x^3)^{-1/2} \cdot (4e^{4x} + 3x^2)$

7. (1 point) Suppose we are given

$$f(x) = \frac{-3 + 6x - x^2}{2x}, \quad f'(x) = \frac{-x^2 + 3}{2x^2}, \quad f''(x) = -\frac{3}{x^3}.$$

Which one of the following statements is true of the value

$$x = -\sqrt{3} \ ?$$

A. It is not a critical point of f, but it is an inflection point.

B. It is neither a critical point nor an inflection point of f.

C. f has a local maximum there, but it is not a global extremum.

D. f has a local and global minimum there.

E. f has a local maximum and a global minimum there.

(F.) f has a local minimum there, but it is not a global minimum.

Your answer: F

$f'(-\sqrt{3}) = 0$

$f''(-\sqrt{3}) = \frac{-3}{-3\sqrt{3}} > 0$

$\therefore$ local min

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{-3+6x-x^2}{2x}$

$= \lim_{x \to \infty} \frac{-3/x + 6 - x}{2} = -\infty$

$\therefore$ not a global min.

8. (1+3+1=5 points) In this question, you will use the definition of the derivative to find the derivative of a function, and then you will check your answer using the quotient rule.

(a) Give the definition of the derivative of a differentiable function $f(x)$ at a point x in its domain.

Answer: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

(b) Using the definition of the derivative, find the derivative of $f(x) = \frac{3x}{4-x}$ at a point x in its domain.

$$f'(x) = \lim_{h \to 0} \frac{\left(\frac{3(x+h)}{4-(x+h)} - \frac{3x}{4-x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3(x+h)(4-x) - 3x(4-x-h)}{(4-h-x)(4-x)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3(4x+4h-x^2-xh) - 3(4x-x^2-xh)}{(4-h-x)(4-x)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{12h}{(4-h-x)(4-x)}$$

$$= \lim_{h \to 0} \frac{12}{(4-h-x)(4-x)}$$

$$= \frac{12}{(4-x)^2}$$

(c) Compute the derivative of $f(x) = \frac{3x}{4-x}$ using the quotient rule.

$$f'(x) = \frac{(4-x)3 - 3x(-1)}{(4-x)^2} = \frac{12-3x+3x}{(4-x)^2} = \frac{12}{(4-x)^2}$$

9. (2 + 2 + 1 = 5 points) A population of snails for a nice restaurant grows at a logistic rate, and is harvested regularly. Denoting the fraction of snails harvested each month by the parameter h, the DTDS governing the growth of this population is given by

$$N_{t+1} = N_t(2.2 - N_t) - hN_t,$$

where t is in months, N_t is the population (in thousands of snails) at time t, and $0 \leq h \leq 1$.

(a) Find the equilibria N^* of this system, showing your work below. Your answers may be formulas using the parameter h.

$N^* =$ 0 and $N^* =$ $1.2-h$

$f(N) = N(2.2-N) - hN$

$f(N) = N \iff 2.2N - N^2 - hN = N$

$\iff 2.2N - N - hN = N^2$

$\iff N = 0$ or $2.2 - 1 - h = N$

$\iff N = 1.2 - h$

If you were unable to answer (a), use the (incorrect) formula $\frac{4}{17}(4 - 3h)$ for N^* in parts (b) and (c).

(b) The *positive* equilibrium N^* is stable, and the *equilibrium harvest* S is $S = hN^*$. Using your formula in (a), find the value of h that maximizes S, justifying your answer with Calculus.

Optimal rate of harvest h: 0.6

$S = h(1.2-h) = 1.2h - h^2$

$S' = 1.2 - 2h$

$\therefore S' = 0$ when $1.2 = 2h \iff h = 0.6$

$S'' = -2 < 0$ $\therefore$ S is concave down on its domain

$\therefore h = 0.6$ gives a local and global maximum of S.

(c) Find the maximum value of the equilibrium harvest S that the restaurant can expect.

Maximum value of S: 0.36

$S = h(1.2-h) = 0.6(0.6) = 0.36$

10. (8 points) We would like to sketch the graph of $y = f(x)$. We have computed for you:

$$f(x) = \frac{-x}{x^3-1}, \quad f'(x) = \frac{2x^3+1}{(x^3-1)^2} \quad \text{and} \quad f''(x) = \frac{-6x^2(x^3+2)}{(x^3-1)^3}.$$

Fill in the answers below and complete the following tables and then sketch the graph of the function using this information. **Round your answers to three decimal places.**

$\lim_{x\to 1^+} f(x) =$ $-\infty$ and $\lim_{x\to 1^-} f(x) =$ ∞

$\lim_{x\to -\infty} f(x) =$ 0 and $\lim_{x\to \infty} f(x) =$ 0

$2x^3+1=0$

$\Rightarrow x^3 = -1/2$

$\Rightarrow x = \sqrt[3]{-1/2}$

Critical points of f: $\sqrt[3]{\frac{-1}{2}} \sim -0.794$, 1 ← vertical asymptote

Intervals of x	$x < \sqrt[3]{\frac{-1}{2}}$	$\sqrt[3]{\frac{-1}{2}} < x < 1$	$x > 1$
Sign of $f'(x)$	−	+	+
Behaviour of $y = f(x)$	decreasing	increasing	increasing

$-6x^2(x^3+2)=0$

$\Rightarrow x = 0$ or $x^3 = -2$

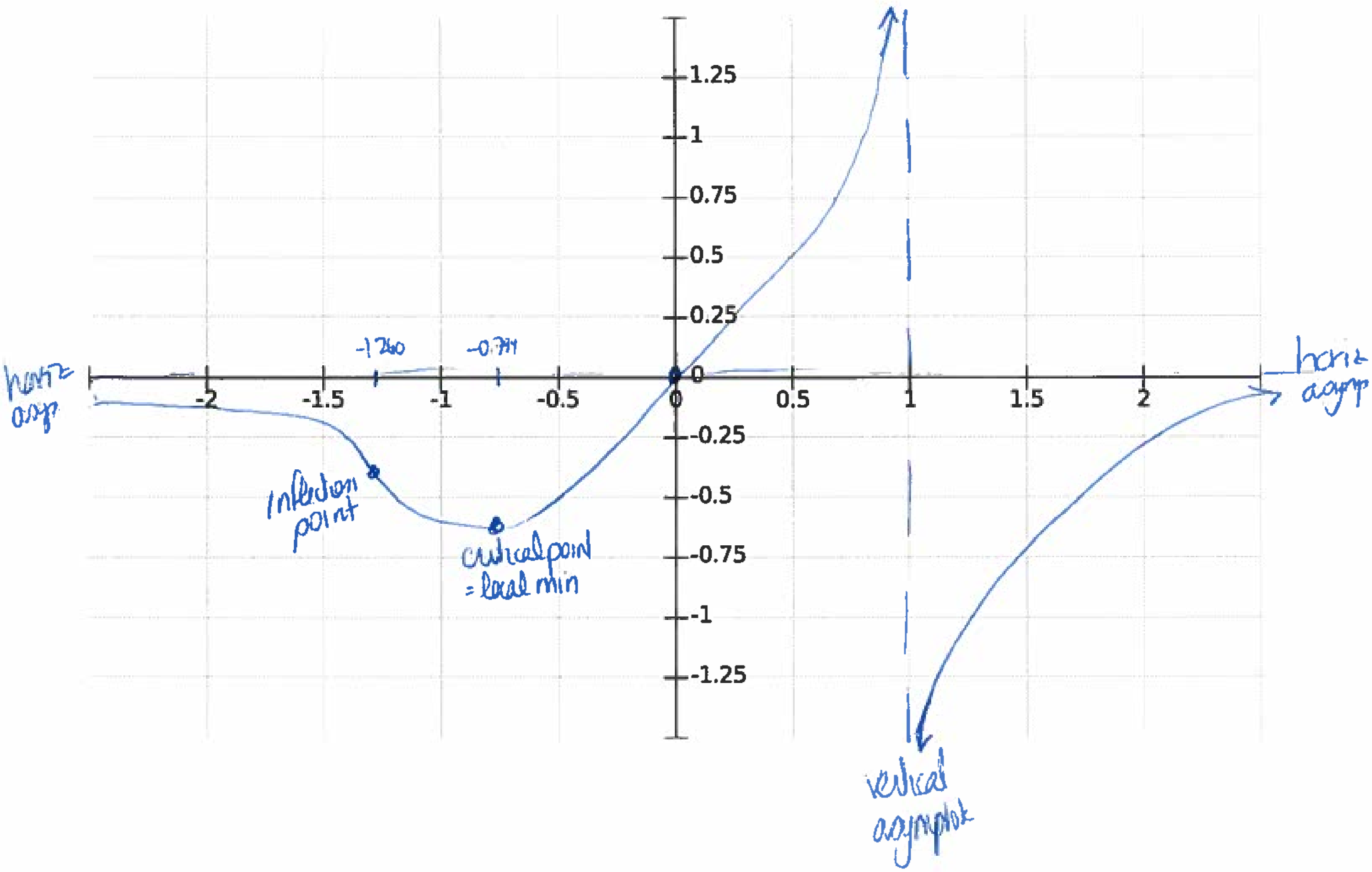
$\Rightarrow x = \sqrt[3]{-2}$

x-values where concavity of f could change: $\sqrt[3]{-2}$ (≈ -1.260), 0, 1

Intervals of x	$x < \sqrt[3]{-2}$	$\sqrt[3]{-2} < x < 0$	$0 < x < 1$	$x > 1$
Sign of $f''(x)$	−	+	+	−
Behaviour of $y = f(x)$	concave down	concave up	concave up	concave down

On the following page, sketch a graph of the function $y = f(x)$. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**

On this page, sketch a graph of the function $y = f(x)$ from Question 10. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**



$f(0)=0$

$f(-0.794) \sim -0.529$

$f(-1.260) \sim -0.420$

1. **(1 point)** If $f(x) = e^x \arctan(x)$ then $f'(1) =$

A. 0
B. $e\pi/4$
C. $e\pi/4 + e/2$
D. $e/2 - e\pi$
E. 1
F. e

Your answer: C

$f'(x) = e^x \arctan(x) + e^x \frac{1}{1+x^2}$

$f'(1) = e^1 \arctan(1) + e^1 \frac{1}{1+1^2}$

$= e\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$\arctan(1) = \theta$

$\Leftrightarrow \tan\theta = 1$ & $-\pi/2 < \theta < \pi/2$

$\Leftrightarrow \theta = \pi/4$

2. **(1 point)** Which of the following is equal to the derivative of $f(x) = 2^{\sin(x)}$?

A. $2^{\sin(x)} \cos(x)$
B. $\ln(2)2^{\sin(x)} \cos(x)$
C. $2^{\cos(x)} \sin(x) \ln(2)$
D. $\frac{2^{\sin(x)}}{\ln(2)}$
E. $2^{\sin(x)}$
F. $\sin(x)\cos(x)2^{\sin(x)-1}$

Your answer: B

3. **(1 point)** Which of the following is equal to the derivative of $g(x) = \ln(e^x x^3)$?

A. $\frac{e^x}{x^3}$
B. $e^x \ln(x^3) - 3e^x/x$
C. $3x^2e^x + x^3e^x$
D. $(3x^2 + x^3)e^x \ln(e^x x^3)$
E. $1 + 3/x$
F. $3/x$

Your answer: E

$g(x) = \ln(e^x x^3)$

$= \ln(e^x) + \ln(x^3)$

$= x + 3\ln(x)$

$g'(x) = 1 + \frac{3}{x}$

4. (1 point) Suppose f is a function and we know that

$$\tan(f(x)) + f(x) = x^2$$

for all x in the domain of f. Then

A. $f'(x) = \dfrac{2x}{\sec(x)\tan(x)+1}$

B. $f'(x) = 2x - \sec(x)\tan(x)$

C. $f'(x) = \dfrac{2x}{\sec^2(x)+1}$

D. $f'(x) = \dfrac{2x}{\sec(f(x))\tan(f(x))+1}$

E. $f'(x) = 2x - \sec^2(f(x))$

(F) $f'(x) = \dfrac{2x}{\sec^2(f(x))+1}$

Your answer: F

$\sec^2(f(x))f'(x) + f'(x) = 2x$

$f'(x)(\sec^2(f(x))+1) = 2x$

$f'(x) = \dfrac{2x}{\sec^2(f(x))+1}$

5. (1 point) Which of the following is the equation of the tangent line to the curve

$$y = x^{1/3} - 16/x$$

at the point $(8, 0)$?

A. $y = \frac{16}{3}x - \frac{128}{3}$

(B.) $y = \frac{1}{3}x - \frac{8}{3}$

C. $y = \frac{2}{3}x - \frac{16}{3}$

D. $y = \frac{1}{3}x^{1/3} + \frac{16}{x} - \frac{8}{3}$

E. $y = -\frac{2}{3}x + \frac{16}{3}$

F. $y = x - 8$

Your answer: B

$y' = \frac{1}{3}x^{-2/3} + 16x^{-2}$

at $x=8$: $y' = \frac{1}{3}\left(\frac{1}{2^2}\right) + \frac{16}{64} = \frac{1}{12} + \frac{1}{4} = \frac{1}{12} + \frac{3}{12} = \frac{1}{3}$

~~at~~ $y = \frac{1}{3}x + b$ at $(8,0) \Rightarrow b = 0 - \frac{1}{3}(8) = -\frac{8}{3}$

6. (1 point) If $h(x) = \sqrt{e^{2x} + x^3}$ then

A. $h'(x) = e^x + \frac{3}{2}\sqrt{x}$

B. $h'(x) = \dfrac{2e^{2x} + 3x^2}{2(e^{2x} + x^3)}$

(C.) $h'(x) = \dfrac{2e^{2x} + 3x^2}{2\sqrt{e^{2x} + x^3}}$

D. $h'(x) = \dfrac{1}{2\sqrt{e^{2x} + x^3}}$

E. $h'(x) = \dfrac{1}{2\sqrt{2e^{2x} + 3x^2}}$

F. $h'(x) = \dfrac{2e^{2x} + 3x^2}{2\sqrt{x}}$

Your answer: C

$h'(x) = \frac{1}{2}(e^{2x} + x^3)^{-1/2}(2e^{2x} + 3x^2)$

7. (1 point) Suppose we are given

$$f(x) = \frac{-3 + 6x - x^2}{x}, \quad f'(x) = \frac{-x^2 + 3}{x^2}, \quad f''(x) = -\frac{6}{x^3}.$$

Which one of the following statements is true of the value

$$x = -\sqrt{3}\ ?$$

A. It is not a critical point of f, but it is an inflection point.

(B.) f has a local minimum there, but it is not a global minimum.

C. f has a local maximum there, but it is not a global extremum.

D. f has a local and global minimum there.

E. f has a local maximum and a global minimum there.

F. It is neither a critical point nor an inflection point of f.

Your answer: B

$f'(-\sqrt{3}) = 0$

$f''(-\sqrt{3}) = \frac{-6}{-3\sqrt{3}} > 0$

$\therefore$ local min

$f(-\sqrt{3}) \doteq 9.46$

$f(1) = 2 < f(-\sqrt{3})$

$\therefore$ not a global min.

8. (1+3+1=5 points) In this question, you will use the definition of the derivative to find the derivative of a function, and then you will check your answer using the quotient rule.

(a) Give the definition of the derivative of a differentiable function $f(x)$ at a point x in its domain.

Answer: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(b) Using the definition of the derivative, find the derivative of $f(x) = \dfrac{2x}{3-x}$ at a point x in its domain.

$$f'(x) = \lim_{h \to 0} \frac{\left(\frac{2(x+h)}{3-(x+h)} - \frac{2x}{3-x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{(2x+2h)(3-x) - 2x(3-x-h)}{(3-x)(3-x-h)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{6x - 2x^2 + 6h - 2hx - 6x + 2x^2 + 2xh}{(3-x)(3-x-h)}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{6h}{(3-x)(3-x-h)}\right)$$

$$= \lim_{h \to 0} \frac{6}{(3-x)(3-x-h)}$$

$$= \frac{6}{(3-x)^2}$$

(c) Compute the derivative of $f(x) = \dfrac{2x}{3-x}$ using the quotient rule.

$$f'(x) = \frac{(3-x)(2) - (2x)(-1)}{(3-x)^2} = \frac{6-2x+2x}{(3-x)^2} = \frac{6}{(3-x)^2}$$

same answer ✓

9. (2 + 2 + 1 = 5 points) A population of snails for a nice restaurant grows at a logistic rate, and is harvested regularly. Denoting the fraction of snails harvested each month by the parameter h, the DTDS governing the growth of this population is given by

$$N_{t+1} = N_t(4.2 - N_t) - hN_t,$$

where t is in months, N_t is the population (in thousands of snails) at time t, and $0 \leq h \leq 2$.

(a) Find the equilibria N^* of this system, showing your work below. Your answers may be formulas using the parameter h.

$N^* =$ 0 and $N^* =$ 3.2 − h $f(N) = N(4.2-N) - hN$

$N = f(N) = 4.2N - N^2 - hN$

$\Leftrightarrow N^2 = 3.2N - hN$

$\Leftrightarrow N = 0$ or $N = 3.2 - h$

If you were unable to answer (a), use the (incorrect) formula $\frac{4}{17}(4 - 3h)$ for N^* in parts (b) and (c).

(b) The *positive* equilibrium N^* is stable, and the *equilibrium harvest* S is $S = hN^*$. Using your formula in (a), find the value of h that maximizes S, justifying your answer with Calculus.

Optimal rate of harvest h: 1.6

$S = hN^* = h(3.2-h) = 3.2h - h^2$

$S' = 3.2 - 2h$

$\therefore S' = 0 \Leftrightarrow 2h = 3.2 \Leftrightarrow h = 1.6$

$S'' = -2$ ∴ S is always concave down.

∴ on $[0, 2]$, h = 1.6 is a local and global max.

(c) Find the maximum value of the equilibrium harvest S that the restaurant can expect.

Maximum value of S: 2.56 $S = hN^* = h(3.2-h) = 1.6(1.6)$

10. **(8 points)** We would like to sketch the graph of $y = f(x)$. We have computed for you:

$$f(x) = \frac{x}{x^3 - 1}, \quad f'(x) = \frac{-2x^3 - 1}{(x^3-1)^2} \quad \text{and} \quad f''(x) = \frac{6x^2(x^3+2)}{(x^3-1)^3}.$$

Fill in the answers below and complete the following tables and then sketch the graph of the function using this information. **Round your answers to three decimal places.**

$\lim_{x\to 1^+} f(x) =$ ∞ and $\lim_{x\to 1^-} f(x) =$ $-\infty$ since "$\frac{1}{0^+} = \infty$" and "$\frac{1}{0^-} = -\infty$"

$\lim_{x\to -\infty} f(x) =$ 0 and $\lim_{x\to\infty} f(x) =$ 0 since $\lim_{x\to\pm\infty} \frac{1/x^3}{1 - 1/x^3} = 0$

$-2x^3 - 1 = 0$
$\Leftrightarrow 2x^3 = -1$
$\Leftrightarrow x^3 = -\frac{1}{\sqrt[3]{2}}$

Critical points of f: $-\frac{1}{\sqrt[3]{2}} \sim -0.794$, 1 ← asymptote.

Intervals of x	$x < -1/\sqrt[3]{2}$	$-1/\sqrt[3]{2} < x < 1$	$x > 1$
Sign of $f'(x)$	$+$	$-$	$-$
Behaviour of $y = f(x)$	increasing	decreasing	decreasing

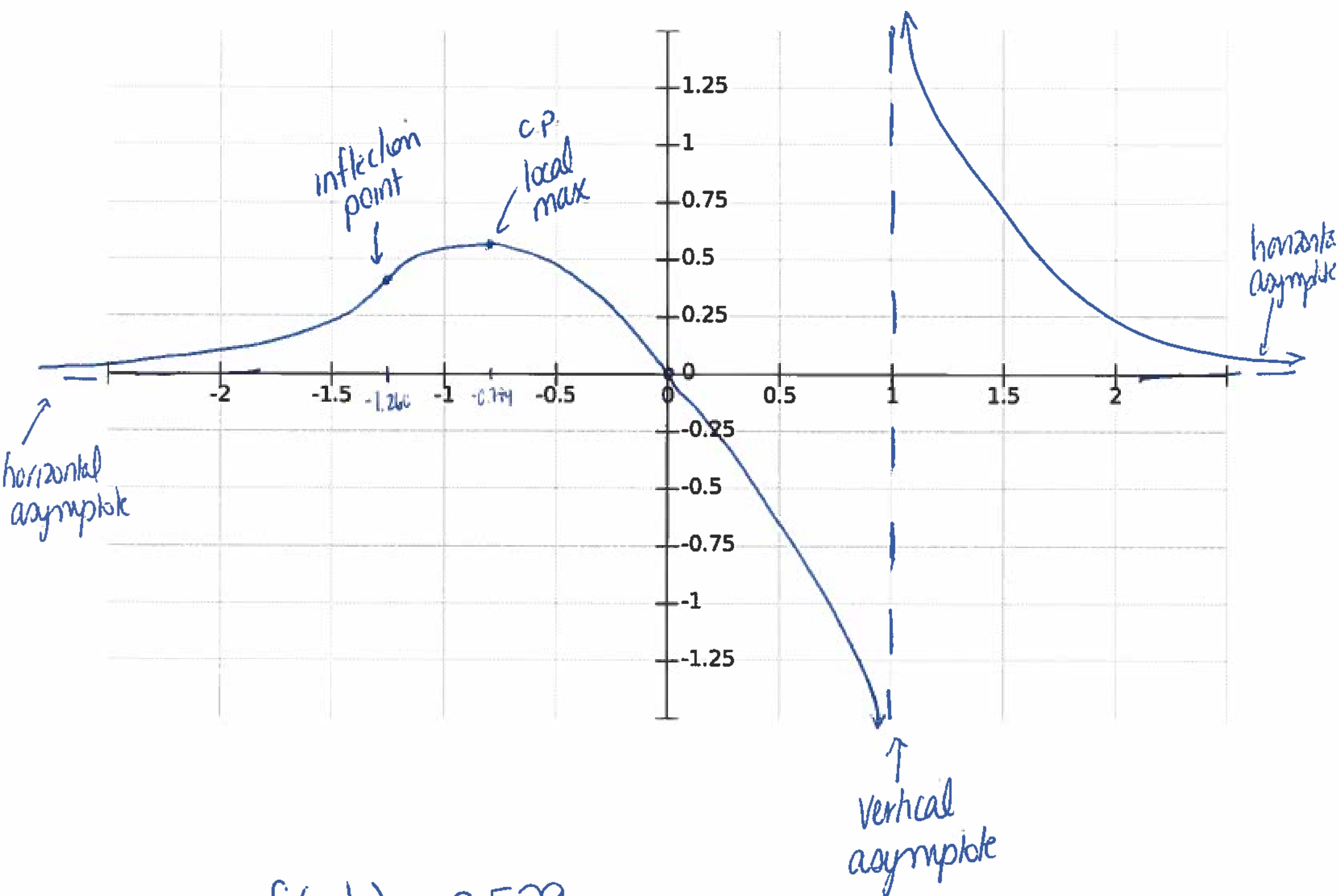
$6x^2(x^3+2) = 0$
$\Leftrightarrow x = 0$ or $x^3 + 2 = 0$
$\Leftrightarrow x = -\sqrt[3]{2} \sim$

x-values where concavity of f could change: $-\sqrt[3]{2} \sim -1.260$, 0 , 1 ← asymptote

Intervals of x	$x < -\sqrt[3]{2}$	$-\sqrt[3]{2} < x < 0$	$0 < x < 1$	$x > 1$
Sign of $f''(x)$	$+$	$-$	$-$	$+$
Behaviour of $y = f(x)$	concave up	concave down	concave down	concave up

On the following page, sketch a graph of the function $y = f(x)$. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**

On this page, sketch a graph of the function $y = f(x)$ from Question 10. **Mark all critical points and inflection points; indicate all asymptotes with a dashed line.**



$f\left(\frac{-1}{\sqrt[3]{2}}\right) = 0.529$

$f(-\sqrt[3]{2}) = 0.420$

$f(0) = 0$