

# Solutions,

## Final Exam Mat2322 C - April 2008

Instructor: Aziz Khanchi

Last Name: ..... First Name: .....

Student number: ..... Seat:.....

### INSTRUCTIONS.

1. Exam period: 3 hours.
2. No books, no notes and no calculators are allowed.
3. There are 10 questions to be answered.
4. You have to justify that your answer is correct.

GOOD LUCK!

Question 1. Consider the function

$$f(x, y) = xy + \frac{8}{x^2} + \frac{8}{y^2}.$$

Find all the critical points and classify them as saddle, maximum or minimum. Notice that the domain of  $f$  does not include  $x = y = 0$ .

$$f(x, y) = xy + 8x^{-2} + 8y^{-2}$$

$$f_x = y - 16x^{-3}, \quad f_y = x - 16y^{-3}. \quad \text{If } \nabla f = \vec{0} \text{ then}$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightarrow \begin{cases} y = \frac{16}{x^3} & \textcircled{1} \\ x = \frac{16}{y^3} & \textcircled{2} \end{cases} \rightarrow x^3 y = 16 \textcircled{3}$$

$$\text{Plug in } \textcircled{3} \text{ into } \textcircled{2} \quad \left(\frac{16}{y^3}\right)^3 y = 16 \rightarrow \frac{16^2}{y^8} = 1 \rightarrow y^8 = 2^8$$

$$\Rightarrow y = \pm 2. \quad \text{if } y = +2 \rightarrow x = 2 \quad \& \quad y = -2 \rightarrow x = -2.$$

The critical points are  $(-2, -2), (2, 2)$ .

$$f_{xx} = 48x^{-4} \quad f_{yy} = 48y^{-4} \quad f_{xy} = 1$$

$$D = 48^2 x^{-4} y^{-4} - 1 \quad \text{at } (-2, -2) \text{ or } (2, 2) \quad D > 0 \text{ Moreover,}$$

$$f_{xx}(-2, -2) = f_{xx}(2, 2) = \frac{48}{(\pm 2)^4} = 3 > 0$$

$\therefore (2, 2), (-2, -2)$  are

local minimums.

Question 2. Use Lagrange Multiplier method to find the local maximum and minimum values of the function

$$f(x, y, z) = x - 2y + 5z$$

subject to the constraint  $g(x, y, z) = x^2 + y^2 + z^2 = 30$ . [Hint: There are only one max and one min.]

$$\text{Solve } \begin{cases} \nabla f = \lambda \nabla g \\ g = 30 \end{cases} \quad \begin{aligned} \nabla f &= \vec{i} - 2\vec{j} + 5\vec{k} \\ \nabla g &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \end{aligned}$$

the first eq. is  $1 = 2\lambda x$ ,  $-2 = 2\lambda y$ ,  $5 = 2\lambda z$

$\therefore x = \frac{1}{2\lambda}$ ,  $y = \frac{-1}{\lambda}$ ,  $z = \frac{5}{2\lambda}$  plug in these in  $g = 30$ .

$$30 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = \left(\frac{1}{4} + \frac{4}{4} + \frac{25}{4}\right) \frac{1}{\lambda^2}$$

$$4\lambda^2 = 1 \rightarrow \lambda = \pm \frac{1}{2}$$

$$\text{if } \lambda = \frac{1}{2} \rightarrow x = 1, y = -2, z = 5$$

$$\text{if } \lambda = -\frac{1}{2} \rightarrow x = -1, y = 2, z = -5$$

$$f(1, -2, 5) = 1 - 2(-2) + 5(5) = 30 \quad \leftarrow \text{local max}$$

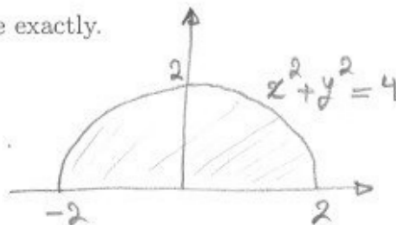
$$f(-1, 2, -5) = -1 - 2(2) + 5(-5) = -30 \quad \leftarrow \text{local min}$$

Question 3. Convert the integral

$$\int_{-2}^2 \int_0^{\sqrt{4-x^2}} e^{-(x^2+y^2)} dy dx,$$

to polar coordinates and evaluate exactly.

The region is



$$\int_{\theta=0}^{\pi} \int_0^2 e^{-r^2} r dr d\theta = \int_0^{\pi} \left. \frac{-1}{2} e^{-r^2} \right|_0^2 d\theta =$$

$$\frac{\pi}{2} (1 - e^{-4}).$$

Question 4. Find the mass of the solid which is bounded between the two spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 9$  with density function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2}.$$

Using spherical coordinates:  $0 \leq \phi \leq \pi$

$$0 \leq \theta \leq 2\pi$$

radius of inner sphere  $\rightarrow 1 \leq \rho \leq 3 \leftarrow$  radius of outer sphere.

$$\text{Mass} = \int_W \frac{1}{x^2 + y^2 + z^2} dV = \int_0^{2\pi} \int_0^{\pi} \int_1^3 \frac{1}{\rho^2} (\rho^2 \sin \phi) d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (3-1) \sin \phi d\phi d\theta =$$

$$2 \int_0^{2\pi} [-\cos \phi]_0^{\pi} d\theta = 2(+1+1) \int_0^{2\pi} d\theta = 8\pi.$$

Question 5. Calculate  $\int_C \vec{F} \cdot d\vec{r}$  when

$$\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + \vec{k},$$

and  $C$  is the line from the origin to the point  $(3, 3, 3)$ .

A parametrization of the line is

$$\vec{r}(t) = \underbrace{(0, 0, 0)}_{\substack{\text{initial} \\ \text{point}}} + \underbrace{(3, 3, 3)}_{\substack{(3, 3, 3) - (0, 0, 0) \\ \text{parallel to the} \\ \text{line}}} t = 3t\vec{i} + 3t\vec{j} + 3t\vec{k}$$

where  $0 \leq t \leq 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^1 \vec{F}(3t, 3t, 3t) \cdot \vec{r}'(t) dt$$

$$[6t\vec{i} + 6t\vec{j} + \vec{k}] \cdot [3\vec{i} + 3\vec{j} + 3\vec{k}] = 18t + 18t + 3 = 36t + 3$$

$$\text{So } \int_C \vec{F} \cdot d\vec{r} = 3 \int_0^1 (12t + 1) dt = 3 (6t^2 + t) \Big|_0^1 = 3(6 + 1) = 21.$$

Question 6. Given that

$$\vec{G} = 3x \vec{i} + y^2 \vec{j},$$

compute  $\int_C \vec{G} \cdot d\vec{r}$ , where  $C$  is part of a hyperbola beginning at the point  $(1, 4)$  and ending at the point  $(0, 5)$ . (Messy calculation can be avoided.)

Notice that  $\text{curl } \vec{G} = \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(3x) = 0 - 0 = 0$  &  
 $\vec{G}$  is defined everywhere

$\therefore \vec{G}$  is a gradient field.  $f$  exists s.t.  $\vec{G} = \nabla f$ .

$$\begin{cases} f_x = 3x \\ f_y = y^2 \end{cases} \rightarrow f = \frac{3}{2}x^2 + C(y) \rightarrow f_y = C'(y)$$

So from the 2nd. eq.  $C'(y) = y^2 \rightarrow C(y) = \frac{1}{3}y^3$

$$\Rightarrow f = \frac{3}{2}x^2 + \frac{1}{3}y^3$$

$$\int_C \vec{G} \cdot d\vec{r} = f(0, 5) - f(1, 4) = \frac{5^3}{3} - \left(\frac{3}{2} + \frac{4^3}{3}\right) = \frac{125}{3} - \frac{3}{2} - \frac{64}{3}$$

$$= \frac{61}{3} - \frac{3}{2} = \frac{122-9}{6} = \frac{113}{6}$$

Question 7. A moving particle has position vector

$$\vec{r}(t) = \left(\frac{t^2}{2} - \frac{t^4}{4}\right) \vec{i} + \frac{2t^3}{3} \vec{j} + \vec{k}, \text{ for } -1 \leq t \leq 1.$$

(a) Find the velocity vector of the particle at time  $t$ .

(b) What is the distance traveled by the particle during the interval  $-1 \leq t \leq 1$ ? Use the fact that  $\|\vec{v}(t)\| = \sqrt{t^2(1+t^2)^2}$ .

$$a) \vec{v}(t) = \vec{r}'(t) = (t - t^3) \vec{i} + 2t^2 \vec{j} + 0 \vec{k}$$

$$b) \|\vec{v}(t)\| = \begin{cases} t(1+t^2) & t \geq 0 \\ -t(1+t^2) & t < 0 \end{cases}$$

$$\text{distance traveled} = \int_{t=-1}^{t=1} \|\vec{v}(t)\| dt = \int_{-1}^0 [-t(1+t^2)] dt + \int_0^1 t(1+t^2) dt$$

$$\begin{aligned} &= \left(-\frac{t^2}{2} - \frac{t^4}{4}\right) \Big|_{-1}^0 + \left(\frac{t^2}{2} + \frac{t^4}{4}\right) \Big|_0^1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \frac{2+1+2+1}{4} \\ &= \frac{6}{4} = \frac{3}{2} \end{aligned}$$

Question 8. Compute the flux of the vector field

$$\vec{F} = (x - 2y)\vec{i} + (4y - 2z)\vec{j} + 3x\vec{k},$$

through the surface  $S$ , where  $S$  is the part of the plane  $z = x + 2y$  above the rectangle  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ , oriented upward.

$$z = f(x, y) = x + 2y$$

$$\int_S \vec{F} \cdot d\vec{A} = \int_{xy\text{-region}} \vec{F} \cdot (-f_x \vec{i} - f_y \vec{j} + \vec{k}) dx dy =$$

$$\int_{x=0}^2 \int_{y=0}^3 [(x-2y)\vec{i} + (4y-2z)\vec{j} + 3x\vec{k}] \cdot (-\vec{i} - 2\vec{j} + \vec{k}) dy dx =$$

$$\int_{x=0}^2 \int_{y=0}^3 \underbrace{(2y - x - 8y + 4z + 3x)}_{-6y + 2x + 4(x+2y)} dy dx = \int_0^2 \int_0^3 \underbrace{(2y + 6x)}_{y^2 + 6xy} dy dx = 9 + 18x$$

$$= \int_0^2 [9 + 18x] dx = [9x + 9x^2]_0^2 = 18 + 36 = 54.$$

Question 9. Use the Divergence Theorem to evaluate the flux integral

$$\int_S \underbrace{[x^3 \vec{i} + (y^3 + \cos z) \vec{j} + (z^3 + x^2 + y^2) \vec{k}]}_{\vec{F}} \cdot d\vec{A},$$

where  $S$  is the surface of the cylinder  $x^2 + y^2 = 1$  bounded by the planes  $z = -1$  and  $z = 1$  (including the ends of the cylinder). It may be helpful to use cylindrical coordinates somewhere.

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x} (x^3) + \frac{\partial}{\partial y} (y^3 + \cos z) + \frac{\partial}{\partial z} (z^3 + x^2 + y^2) \\ &= 3x^2 + 3y^2 + 3z^2 \end{aligned}$$

*Cyl. coord.*

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \operatorname{div} \vec{F} \, dV = \int (3r^2 + 3z^2) (r \, dr \, d\theta \, dz)$$

$$= \int_{z=-1}^1 \int_0^1 \int_0^{2\pi} (3r^3 + 3z^2 r) \, d\theta \, dr \, dz =$$

$$6\pi \int_{-1}^1 \int_0^1 (r^3 + z^2 r) \, dr \, dz = 6\pi \int_{-1}^1 \left[ \frac{1}{4} r^4 + \frac{1}{2} z^2 r^2 \right]_0^1 dz$$

$$= 6\pi \left( \frac{1}{4} z + \frac{1}{6} z^3 \right) \Big|_{-1}^1 =$$

$$6\pi \left( \frac{1}{4} + \frac{1}{6} + \frac{1}{4} + \frac{1}{6} \right) =$$

$$6\pi \left( \frac{1}{2} + \frac{1}{3} \right) = 6\pi \frac{3+2}{6}$$

$$= 5\pi$$