Solutions to the Final exam of Winter 2009
Applied Ordinary Differential Equations
ENGR 213

(1) (a) This is a separable equation which can be re-written in the form

$$\frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2}.$$

To integrate the left hand side, use the substitution $u = 2y + 3$, thus $du = 2dy$, while, for the right hand side, use $w = 4x + 5$, $dw = 4dx$. Therefore,

$$\int \frac{dy}{(2y+3)^2} = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} + c = -\frac{1}{2(2y+3)} + c,$$

$$\int \frac{dy}{(4x+5)^2} = \frac{1}{4}\int \frac{1}{w^2}\,dw = -\frac{1}{4w} + c' = -\frac{1}{4(4x+5)} + c'.$$

Using a single constant C to replace the difference $c' - c$, we have the general solution of the equation in implicit form:

$$\frac{1}{2(2y+3)} = \frac{1}{4(4x+5)} + C.$$

(b) This ODE is linear, hence we shall first put it in its standard form (we'll solve it for $x > 0$):

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}e^x.$$

Thus $P(x) = 1/x$ and $Q(x) = e^x/x$ and we look for an integrating factor (hence in the following we choose the constant of integration zero)

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

We multiply the equation with μ to obtain $xy' + y = e^x$ or $(xy)' = e^x$. Integrating both sides, we obtain

$$xy = e^x + C, \quad C = \text{constant}, \quad \Rightarrow \quad y(x) = \frac{e^x}{x} + \frac{C}{x}, \quad C = \text{constant},$$

the general solution in explicit form.

We now make use of the initial condition $y(1) = 2$ to determine the constant:

$$y(1) = e + C = 2 \quad \Rightarrow \quad C = 2 - e \quad \Rightarrow y(x) = \frac{e^x}{x} + \frac{2-e}{x}.$$

(2) By denoting $M(x,y) = y^2\cos x - 3x^2y - 2x$, $\quad N(x,y) = 2y\sin x - x^3 + \ln y$, we check that

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y) = 2y\cos x - 3x^2$$

hence the equation is exact.

We therefore start to look for a function $f(x,y)$ whose partial derivatives with respect to x, respectively y, are M and respectively N. We do so by integrating $M(x,y)$ with respect to x:

$$f(x,y) = y^2 \sin x - x^3 y - x^2 + C(y).$$

We then differentiate f with respect to y and, setting the result equal to N, obtain

$$2y \sin x - x^3 + C'(y) = \ln y.$$

Thus $C(y)$ is an antiderivative of $\ln y$. We can find it by integration by parts ($u = \ln y,\ dv = dy$):

$$C(y) = \int \ln y\, dy = y \ln y - \int 1\, dy = y \ln y - y + c, c = \text{constant}.$$

Hence $f(x,y) = y^2 \sin x - x^3 y - x^2 + y \ln y - y + c$ and the solution to the ODE (in implicit form) is

$$y^2 \sin x - x^3 y - x^2 + y \ln y - y + c = 0, \quad c = \text{constant}.$$

(3) We'll use the substitution $u = x + y \Rightarrow y = u - x$, then $\dfrac{dy}{dx} = \dfrac{du}{dx} - 1$. Therefore, the substitution leads to the related ODE

$$\frac{du}{dx} - 1 = \tan^2 u \quad \Leftrightarrow \quad \frac{du}{dx} = 1 + \tan^2 u.$$

As $\tan u = \dfrac{\sin u}{\cos u}$, we note that $1 + \tan^2 u = \dfrac{\sin^2 u + \cos^2 u}{\cos^2 u} = \dfrac{1}{\cos^2 u}$. Then

$$\frac{du}{dx} = \frac{1}{\cos^2 u} \quad \Rightarrow \quad \cos^2 u\, du = dx$$

whose solution is $\displaystyle\int \frac{1 + \cos(2u)}{2}\, du = x + c \quad \Rightarrow \quad \frac{u}{2} + \frac{\sin(2u)}{4} = x + c$, thus the general solution of the original ODE (in implicit form) is

$$\frac{x+y}{2} + \frac{\sin(2(x+y))}{4} = x + c, \ c = \text{constant}.$$

(4) For the first 4 seconds, the velocity is constant (equal to $100\,\text{m/sec}$), then the velocity is a solution of the ODE

$$\frac{dV}{dt} = -0.002V^2.$$

This ODE has the solution

$$-\frac{1}{V} = -0.002t + c, \quad V(0) = 100$$

or

$$V(t) = \frac{1}{0.002t + 0.01} \quad \Rightarrow \quad \frac{1}{0.002t + 0.01} = 20 \quad \Rightarrow t = 20.$$

We must not forget to add the 4 seconds when the velocity remained constant, hence the final answer is that the velocity reaches 20 m/sec after 24 sec.

(5) (a) First, using $r^2+6r+8=0 \quad\Leftrightarrow\quad r=-4$ or $r=-2$, we find that the complementary part of the general solution is $y_c(x)=c_1e^{-4x}+c_2e^{-2x}$.

Having no duplication between y_c and $\sin(3x)$, we set up y_p as

$$y_p(x)=A\sin(3x)+B\cos(3x)\ \Rightarrow\ y_p'(x)=3A\cos(3x)-3B\sin(3x)\ \Rightarrow\ y_p''(x)=-9A\sin(3x)-9B\cos(3x).$$

Substituting these into the nonhomogenous ODE, we obtain

$$-9A\sin(3x)-9B\cos(3x)+6(3A\cos(3x)-3B\sin(3x))+8(A\sin(3x)+B\cos(3x))=\sin(3x),$$

or $-A-18B=1,\quad 18A-B=0$ which implies $A=-1/325$ and $B=-18/325$.
Thus the general solution of the ODE is

$$y(x)=c_1e^{-4x}+c_2e^{-2x}-\frac{1}{325}\sin(3x)-\frac{18}{325}\cos(3x),\quad c_{1,2}=\text{constants}.$$

(b) We start again with the characteristic equation of the associated homogenous ODE, $r^2+10r+25=0$. This equation has 5 as a double root, hence $y_c(x)=c_1e^{5x}+c_2xe^{5x}$. Yet, there is no duplication between y_c and e^x so we may take $y_p(x)=Ae^x=y_p'(x)=y_p''(x)$. In conclusion, $36Ae^x=e^x$, so $A=1/36$ and the general solution of the ODE is

$$y(x)=c_1e^{5x}+c_2xe^{5x}+\frac{1}{36}e^x,\quad c_{1,2}=\text{constants}.$$

(6) This is a non-homogeneous Cauchy-Euler equation. Consider first the homogeneous Cauchy-Euler DE: $2x^2y''+5xy'+y=0$. Its characteristic equation is $2m(m-1)+5m+1=0$ or $2m^2+3m+1=0$. Its roots are -1 and $-1/2$, hence (solving the ODE on $(0,\infty)$)

$$y_c(x)=c_1\frac{1}{x}+c_2\frac{1}{\sqrt{x}},\quad c_{1,2}=\text{constants}.$$

By choosing, $y_1(x)=x^{-1}$ and $y_2(x)=x^{-1/2}$, we can form the Wronskian $W(x)$ and proceed to find a particular solution of the non-homogeneous equation given by using the variation of parameters. However, in order to apply the variation of parameters and identify correctly the function f needed for W_1 and W_2 below, we must use the standard form of the equation: $y''+P(x)y'+Q(x)=f(x)$, thus divide the non-homogeneous Cauchy-Euler equation by $2x^2$ to obtain

$$y''+\frac{5}{2x}y'+\frac{1}{2x^2}y=\frac{1}{2}-\frac{1}{2x}.$$

$$W(x)=\det\begin{pmatrix}x^{-1} & x^{-1/2}\\ -x^{-2} & -\frac{1}{2}x^{-3/2}\end{pmatrix}=\frac{1}{2}x^{-5/2},\ W_1(x)=\det\begin{pmatrix}0 & x^{-1/2}\\ \frac{1}{2}-\frac{1}{2x} & -\frac{1}{2}x^{-3/2}\end{pmatrix}=-\frac{1}{2}x^{-1/2}+\frac{1}{2}x^{-3/2}$$

$$W_2(x)=\det\begin{pmatrix}x^{-1} & 0\\ -x^{-2} & \frac{1}{2}-\frac{1}{2x}\end{pmatrix}=\frac{1}{2}x^{-1}-\frac{1}{2}x^{-2},$$

$$u_1' = \frac{W_1}{W} = -x^2 + x, \quad u_2' = \frac{W_2}{W} = x^{3/2} - x^{1/2}.$$

The simplest antiderivatives are

$$u_1 = -\frac{x^3}{3} + \frac{x^2}{2}, \quad u_2 = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}.$$

Thus

$$y_p(x) = x^{-1}\left(-\frac{x^3}{3} + \frac{x^2}{2}\right) + x^{-1/2}\left(\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right) = \frac{x^2}{15} - \frac{x}{6},$$

and

$$y_{general}(x) = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}} + \frac{x^2}{15} - \frac{x}{6}, \quad c_{1,2} = \text{constants}.$$

(7) We'll use the method of systematic elimination, hence, using D for the differentiation with respect to t, we'll write the system as

$$\begin{cases} Dx &= 2x + 3y - e^{2t}, \\ Dy &= -x - 2y + e^{2t} \end{cases} \quad \text{or} \quad \begin{cases} (D-2)x - 3y &= -e^{2t}, \\ x + (D+2)y &= e^{2t}. \end{cases}$$

Through symbolic computation, we obtain

$$\begin{cases} (D-2)(D+2)x - 3(D+2)y &= -(D+2)e^{2t} \\ 3x + 3(D+2)y &= 3e^{2t} \end{cases} \Rightarrow \begin{cases} (D^2-4)x - 3(D+2)y &= -4e^{2t} \\ 3x + 3(D+2)y &= 3e^{2t}. \end{cases}$$

Above, we used that $(D+2)e^{2t} = (e^{2t})' + 2e^{2t} = 4e^{2t}$. Adding the two equations, we obtain

$$(D^2 - 1)x = -e^{2t} \quad \Leftrightarrow \quad x'' - 1 = -e^{2t}.$$

We solve first this non-homogeneous ODE. It's easy to see that

$$x_c(t) = c_1 e^t + c_2 e^{-t}, \quad c_{1,2} = \text{constants}.$$

For the particular solution, we use the method of undetermined coefficients by setting $x_p(t) = Ae^{2t} \Rightarrow x_p'(t) = 2Ae^{2t}$, $x_p''(t) = 4Ae^{2t}$, thus $3A = -1$ and

$$x_p(t) = -\frac{1}{3}\,e^{2t} \quad \Rightarrow \quad x(t) = c_1 e^t + c_2 e^{-t} - \frac{1}{3}\,e^{2t}, \quad c_{1,2} = \text{constants}.$$

To find $y(t)$ we go back to the first equation of the system and note that

$$y = \frac{1}{3}\left(x' - 2x + e^{2t}\right) = -\frac{c_1}{3}e^t - c_2 e^{-t} + \frac{1}{3}e^{2t}.$$

In conclusion, the general solution of the given system is

$$\begin{cases} x(t) &= c_1 e^t + c_2 e^{-t} - \frac{1}{3}\,e^{2t} \\ y(t) &= \frac{1}{3}\left(x' - 2x + e^{2t}\right) = -\frac{c_1}{3}e^t - c_2 e^{-t} + \frac{1}{3}e^{2t} \end{cases} \quad c_{1,2} = \text{constants}.$$

(8) Assume that the ODE has a solution of the form $y(x) = \sum_{n\geq 0} a_n x^n$. Then $y'(x) = \sum_{n\geq 1} n a_n x^{n-1}$ and $y''(x) = \sum_{n\geq 2} n(n-1)a_n x^{n-2}$. As y is a solution, we must have

$$\sum_{n\geq 2} n(n-1)a_n x^{n-2} - 3x\sum_{n\geq 1} n a_n x^{n-1} - \sum_{n\geq 0} a_n x^n = 0.$$

Equivalently,

$$\sum_{n\geq 2} n(n-1)a_n x^{n-2} - \sum_{n\geq 1} 3n a_n x^n - \sum_{n\geq 0} a_n x^n = 0.$$

We'll re-index the first sum as follows

$$\sum_{n\geq 2} n(n-1)a_n x^{n-2} = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n.$$

Then

$$\sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n - \sum_{n\geq 1} 3n a_n x^n - \sum_{n\geq 0} a_n x^n = 0$$

or

$$2a_2 - a_0 + \sum_{n\geq 1}\left((n+2)(n+1)a_{n+2} - 3n a_n - a_n\right) x^n = 0.$$

Thus $a_2 = a_0/2$ and, generally, for $n \geq 1$,

$$a_{n+2} = \frac{(3n+1)a_n}{(n+2)(n+1)}.$$

Evaluating, we obtain

$$a_3 = \frac{4}{3\cdot 2}\, a_1$$

$$a_4 = \frac{7}{4\cdot 3}\, a_2 = \frac{7}{4\cdot 3\cdot 2}\, a_0$$

and so on.

Thus

$$y(x) = a_0\left(1 + \frac{1}{2}\, x^2 + \frac{7}{24}\, x^4 + \dots\right) + a_1\left(x + \frac{2}{3}\, x^3 + \dots\right)$$

and it remains to find a_0 and a_1 such that $y(0) = 1,\ y'(0) = 0$. Actually, $y(0) = 1 \Rightarrow a_0 = 1$, while $y'(0) = 0 \Rightarrow a_1 = 0$, hence the solution is

$$y(x) = 1 + \frac{1}{2}\, x^2 + \frac{7}{24}\, x^4 + \dots.$$

(9) Note that this problem has been also solved in class so you may also refer to your lecture notes.

(a) The ODE is

$$q'' + 6q' + 18q = 30\cos t,$$

thus

$$q_c(t) = c_1 e^{-3t}\cos(3t) + c_2 e^{-3t}\sin(3t), \quad c_{1,2} = \text{constant}.$$

Setting up $q_p(t) = A\cos t + B\sin t$, we obtain

$$(-A + 6B + 18A)\cos t + (-B - 6A + 18B)\sin t = 30\cos t \quad \Rightarrow$$

$$B = 6A/17 \quad A = 510/325 = 102/65 \quad \Rightarrow \quad B = 36/65.$$

Thus the general solution is

$$q(t) = c_1 e^{-3t}\cos(3t) + c_2 e^{-3t}\sin(3t) + \frac{102}{65}\cos t + \frac{36}{65}\sin t, \quad c_{1,2} = \text{constant}.$$

We see that

$$q'(t) = c_1(-3e^{-3t}\cos(3t) - 3e^{-3t}\sin(3t)) + c_2(-3e^{-3t}\sin(3t) + 3e^{-3t}\cos(3t)) - \frac{102}{65}\sin t + \frac{36}{65}\cos t,$$

so we can now use the initial conditions: $q(0) = 100 = c_1 + 102/65$ and $q'(0) = 0 = -3c_1 + 3c_2 + 36/65 = 0$. Thus

$$c_1 = \frac{6398}{65}, \quad c_2 = \frac{6386}{65},$$

so

$$q(t) = \frac{6398}{65}e^{-3t}\cos(3t) + \frac{6386}{65}e^{-3t}\sin(3t) + \frac{102}{65}\cos t + \frac{36}{65}\sin t.$$

(b) The transient terms are

$$\frac{6398}{65}e^{-3t}\cos(3t) \quad \text{and} \quad \frac{6386}{65}e^{-3t}\sin(3t)$$

(as they go to zero as t approaches infinity) and the steady-state terms are the remaining (last) two. The circuit is underdamped.

$\square$