Family Name_____ First Name_____

Student #_____

SYS 5130 Midterm--October 14, 19:00---20:30

Closed Book

(In total you have 8 questions, 20 marks in total.)

1. (2 points) True/False:

(a) Any optimization problem has at least one variable. _____T_____

(b) Optimization problem may have no constraint. _____T_____

2. (2 points) Consider the following two systems:

System 1: $x+y \geq 10$, $y+z \geq 10$;
System 2: $x+y \geq 10$, $x+2y+z \geq 20$.

Are they equivalent? If yes, explain; if no, construct a counterexample.

Answer: No. For example, (x, y, z)=(9,4,4) is a solution of System 2, but not a solution of System 1, since $y+z=8 < 10$.

3. (2 points) Consider the following system:

$$-3x1 - 2x2 + x3 + x4 - 3x5 \leq 0,$$
$$2x2 - 5x3 + x4 - 4x5 \geq 3,$$
$$5x1 - x3 = 0,$$

where x3 appears three times. Can we find an equivalent system in which x3 appears only once? If no, explain; if yes, find it.

Solution: YES! The equivalent system is:

$$2x1 - 2x2 + x4 \leq 0,$$
$$-25x1 + 2x2 + x4 \geq 3,$$
$$5x1 - x3 = 0$$

4. (2 points) Consider the following system:

Maximize $7x_1 + 9x_2$

Subject to $x_1 + x_2 \le 6$

$5x_1 + 9x_2 \le 44$

$x_1, x_2 \ge 0$ integer.

To find Gomory cuts, we change the two inequalities as:

$x_1 + x_2 + s_1 = 6$

$5x_1 + 9x_2 + s_2 = 44$

$s_1, s_2 \ge 0$ integer.

By eliminating x_2, we obtain

$$4x_1 + 9s_1 - s_2 = 10 \qquad (1)$$

Question: Find the Gomory cut from (1).

Solution: We write (1) as:

$$x_1 + 2s_1 - s_2 - 2 = \frac{2}{4} - \frac{1}{4}s_1 - \frac{3}{4}s_2$$

Note that the left hand side consist only of integers, so the right hand side must add up to an integer. The right hand sides can only be

$$0, -1, -2, \ldots$$

Therefore, we have derived the following two constraints:

$$\frac{2}{4} - \frac{1}{4}s_1 - \frac{3}{4}s_2 \le 0$$

Changing coefficients to integers,

$$s_1 + 3s_2 \ge 2 .$$

This is a Gomory cut.

5. (3 points) Use Fourier Motzkin Method (not software) to solve the following question:

$$
\begin{aligned}
\text{Max} \quad & 5x + y \\
\text{s.t.} \quad & 5x + 2y \le 14 \\
& y \ge 3 \\
& x + 3y \le 13 \\
& x, y \ge 0
\end{aligned}
$$

Step 1: Let $z=5x+y$.

Step 2: Eliminate x:

$$
\begin{aligned}
z + y &\le 14 \\
y &\ge 3 \\
z + 14y &\le 65 \\
z, y &\ge 0
\end{aligned}
\qquad \text{i.e.,} \qquad
\begin{aligned}
y &\le 14 - z \\
y &\ge 3 \\
14y &\le 65 - z \\
z, y &\ge 0
\end{aligned}
$$

Step 3: Eliminate y:

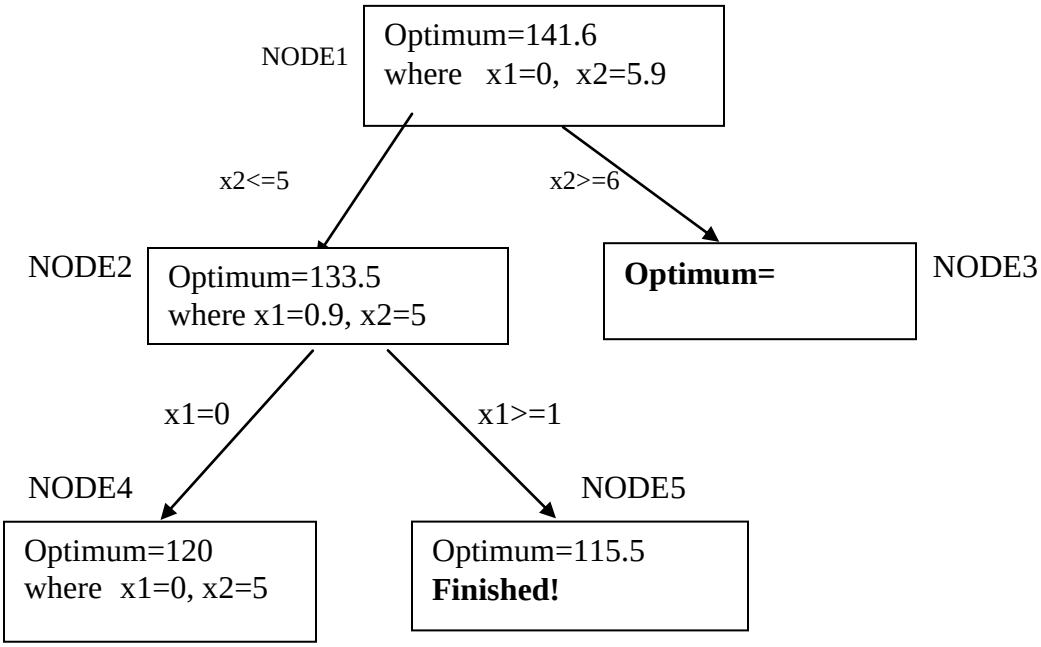
$$
\begin{aligned}
3 &\le 14 - z \\
52 &\le 65 - z \\
&\rightarrow \\
z &\le 11 \\
z &\le 13
\end{aligned}
$$

Step 4: max z=min{11, 13}=11, imply also that y=2, x=1.8.

6. (2 points) We are going to solve the following system using branch and bound method:

$$\begin{aligned} \text{Max} \quad & 15x_1+24x_2 \\ \text{s.t.} \quad & 18x_1+10x_2 \le 90 \\ & 20x_1+20x_2 \le 118 \\ & x_1, \quad x_2 \ge 0 \text{ integer} \end{aligned}$$

Look at the following branch and bound:



(i) For **NODE3**, after running Fourier.exe, we get the following:

x3>=101
x3>=140
x3<=216
x3<=167.4333333
x3<=141.6
x3<=169.5
x3<=142.5

What will be the Optimum?

141.6

(ii) Why we can say "finished" in the Node 5? Explain.

Since the optimal value 115.5 is less than 120 in the Node 4.

7. (3 points) Suppose we have three projects: F, M, D. The value 0 means not selecting the project, and 1 means selecting. Write down logic conditions to each of the following:

(i) At least one project is undertaken;
(ii) If you make the project M, then you can not make the project D; and if you make the project D, then you can not make the project M;
(iii) Project M cannot be undertaken without first completing the project F.

(i) $F+M+D >= 1$

(ii) $M+D \leq 1$

(iii) $M \leq F$.

8. (4 points) A bank makes *four kinds of loans* L1, L2, L3 and L4 to its personal customers. These loans yield the following annual interest rates to the bank:

- L1: 6%,
- L2: 5%,
- L3: 5%,
- L4: 3%.

The bank has a maximum lending capability of 250 thousand dollars and is further restricted by the following policies:

- L3 must be at most 55% of all loans issued, and at least 15% of all loans issued;
- L4 cannot exceed 25% of all loans issued.

Formulate the bank's loan problem as a linear programming so as to maximize interest income while satisfying all the restrictions.
(**Remark. Do not simplify them**)

Solution: Let

x_i (or Li) = the amount to the Li loan in dollars, $1 \le i \le 4$.

Then we have the following linear system:

$$\begin{aligned} \text{Maximize} \quad & 0.06x_1 + 0.05x_2 + 0.05x_3 + 0.03x_4 \\ \text{Subject to} \quad & x_1 + x_2 + x_3 + x_4 \le 250 \\ & x_3 - 0.55(x_1 + x_2 + x_3 + x_4) \le 0 \\ & x_3 - 0.15(x_1 + x_2 + x_3 + x_4) \ge 0 \\ & x_4 - 0.25(x_1 + x_2 + x_3 + x_4) \le 0 \\ & x_i \ge 0 \ (i=1,2,3,4). \end{aligned}$$