13. Optimization (Nov. 1)

Lec 12 mini review.

using f	**find domain**	**find x- and y-intercept(s)**	**find asymptotes: horizontal/vertical**
using f'	**find critical numbers**	**determine behaviour: increasing/decreasing**	**classify extrema: local min/max**
using f''	**find IP candidates**	**determine curvature: concave up/concave down**	**locate inflection points**
using all	**sketch the graph of f**	**label**: intercepts, asymptotes,	extrema, and inflection points

Extreme Values

A function $y = f(x)$ has...

...an **absolute/global maximum at** $x = c$ if

$f(c) \geq f(x)$ for all x in the Domain of f

...an **absolute/global minimum at** $x = c$ if

$f(c) \leq f(x)$ for all x in the Domain of f

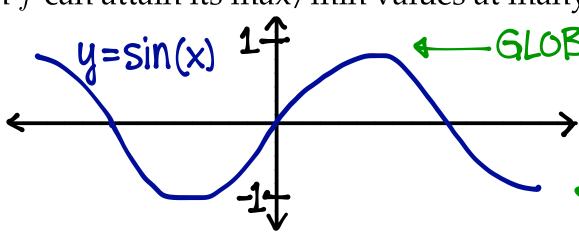
The value $f(c)$ is called the...

absolute/global maximum value of f.

absolute/global minimum value of f.

- ◇ The max/min values of f are called **extreme values**.
- ◇ A function f can attain its max/min values at many numbers.

Ex

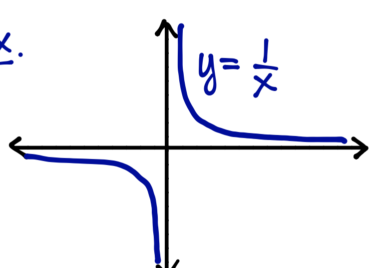


← GLOBAL MAX VALUE is 1
• it's attained by infinitely many #s.

← GLOBAL MIN VALUE is −1
(also attained at infinitely many #s)

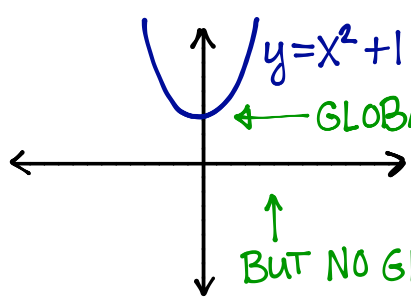
- ◇ Some functions do not have max and/or min values.

Ex.



↑ has NO GLOBAL MIN OR MAX

Ex.



← GLOBAL MIN VALUE is 1
attained at only one number: $x=0$

↑ BUT NO GLOBAL MAX.

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Local Extrema vs. Global Extrema

A function $y = f(x)$ has...

...a **relative/local maximum at** $x = c$ if

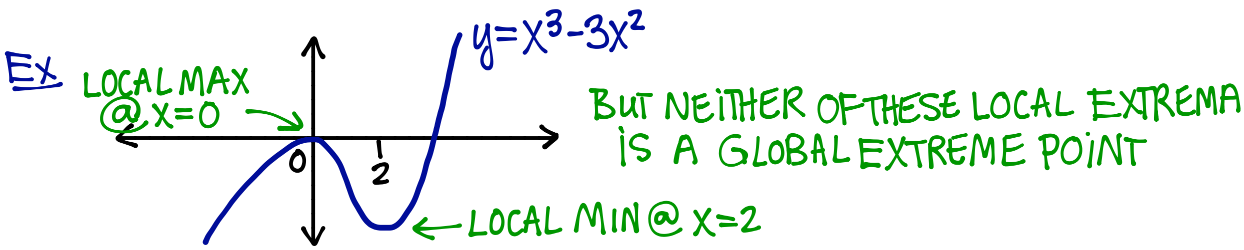
$f(c) \geq f(x)$ for all x "near" c (in an open interval on either side of c)

...a **relative/local minimum at** $x = c$ if

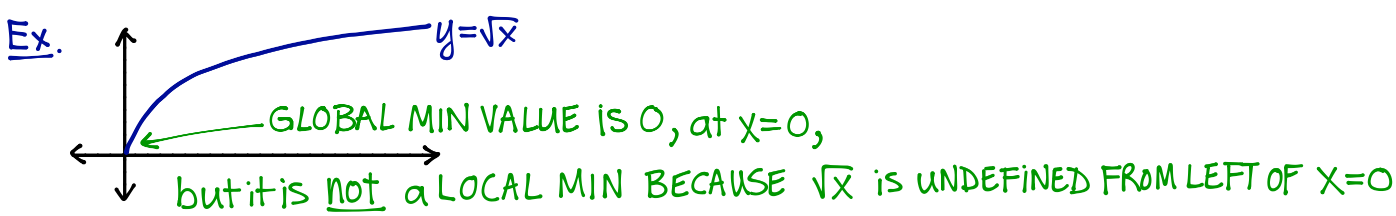
$f(c) \leq f(x)$ for all x "near" c (in an open interval on either side of c)

- ◇ The local max/min values of f are called **local extrema**.
- ◇ Some local max/min are also global extreme points.

- ◇ Not every local max/min is an absolute max/min.

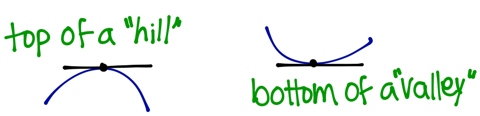


- ◇ Not every global max/min is a local max/min.



Fact. If f has a local min/max at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

In other words, a local max at which the derivative exists is necessarily a "type 1" critical number.

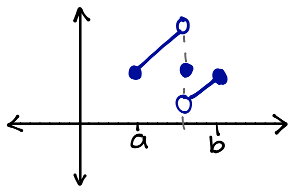


* if f'(c) DNE, then f could still have a LOCAL MIN/MAX at x=c... but it would be at a corner or cusp of f

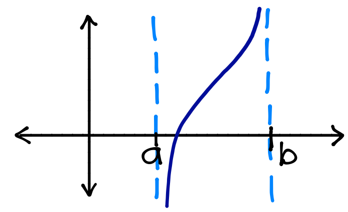
Extreme Value Theorem

Theorem 13.1. (Extreme Value Theorem)
If $y = f(x)$ is continuous on the closed interval $[a, b]$, then, restricted to the interval $[a, b]$, f has an absolute maximum and an absolute minimum on $[a, b]$.

Note. If f is not continuous on $[a, b]$, then the Extreme Value Theorem is not applicable. Even if f is continuous, if the interval is not closed, then the Extreme Value Theorem is not applicable.



this function is NOT continuous on $[a,b]$
It has no GLOBAL EXTREMA on $[a,b]$



this function is continuous on (a,b) but the interval is NOT closed.
It has no GLOBAL EXTREMA on (a,b).

Strategy to find the absolute extrema of a continuous function $f(x)$ on a closed interval $[a, b]$

1. Find the critical numbers of f.
2. For each critical number c such that $c \in [a, b]$, compute its value $f(c)$.
3. For the endpoints $x = a$ and $x = b$ of $[a, b]$, compute the values $f(a)$ and $f(b)$.
4. The absolute maximum value of f on $[a, b]$ is largest value computed in steps 2 and 3.
5. The absolute minimum value of f on $[a, b]$ is smallest value computed in steps 2 and 3.

Example 13.2. Find the extreme values of $g(t) = 2t^{10}(4 - t^2)^5$ on the closed interval $[-1, 2]$.

$g'(t) = 20t^9(4-t^2)^5 + 2t^{10}(5)(4-t^2)^4(-2t)$ ← same domain as g (all real #s) ∴ no type 2 critical #s.

type 1? $0 = g'(t)$

$0 = 20t^9(4-t^2)^5 - 20t^{11}(4-t^2)^4$

$0 = 20t^9(4-t^2)^4[(4-t^2)-t^2]$

biggest common factors we could gather from all terms.

$0 = 20t^9(4-t^2)^4[4-2t^2]$

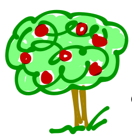
$t=0$ $\quad$ $t^2=4 \Rightarrow t=\pm 2$ $\quad$ $2t^2=4$, $t^2=2$, $t=\pm\sqrt{2}$

1. ∴ the critical #s of g are $t=0, t=-2, t=2, t=-\sqrt{2}, t=\sqrt{2}$
2. values of critical #s on $[-1,2]$: $g(0)=0$, $g(\sqrt{2})=2^{11}=2048$, $g(2)=0$
3. values at endpoints of $[-1,2]$: $g(-1)=486$, $g(2)=0$.
4. g has a GLOBAL MAX on $[-1,2]$ at $x=\sqrt{2}$. GLOBAL MAX VALUE is $g(\sqrt{2})=2^{11}$
5. g has a GLOBAL MIN on $[-1,2]$ at $x=0, x=2$. GLOBAL MIN VALUE is $g(0)=g(2)=0$.

- In Optimization, we are looking for the "best" value of something (or "optimum" value).
- The optimum value is the absolute minimum/maximum, depending on what we're dealing with.
- We know how to find absolute max/min values!
- If we can model our optimization problem as a function of 1 variable, then we know how to find its extreme values.

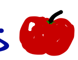
Example 13.3. Assume that the apples in an orchard grow logistically and are harvested according to the following DTDS:

updating function



$$\boxed{x_{t+1} = 2.5x_t(1 - x_t) - hx_t}$$

$f(x) = 2.5x(1-x) - hx$

- the **time step** t is measured in weeks
- the quantity x_t represents the portion of the maximum possible crop size (which has been estimated to be 100 000 kg of apples) available on week t
- the constant parameter $h > 0$ represents the **harvesting intensity** (fraction of available apples harvested each week) (or how greedily we harvest the available apples )

Assume the harvesting intensity h remains fixed throughout the harvest so that the quantity of apples approaches its equilibrium point. Then the equilibrium yield of this crop is $Y(h) = hx^*$, where x^* is the crop's (positive) equilibrium point (fixed point).

(a) Find the fixed point(s) of this system and the corresponding equilibrium yield.

for fixed points (also known as equilibria) we solve $x = f(x)$.

$\Rightarrow x = 2.5x(1-x) - hx$

$\Rightarrow 0 = 2.5x - 2.5x^2 - hx - x$

$\Rightarrow 0 = x(2.5 - 2.5x - h - 1)$

$\Rightarrow 0 = x(1.5 - h - 2.5x)$

$x = 0$ or $1.5 - h - 2.5x = 0$

$\Rightarrow 2.5x = 1.5 - h$

$\Rightarrow x = \dfrac{1.5 - h}{2.5} = 0.6 - 0.4h$

note in order for this to be a positive equilibrium, we need $1.5 - h > 0 \Rightarrow h < 1.5$

∴ this DTDS has 2 equilibria:
$x^* = 0$ and $x^* = 0.6 - 0.4h$

FACT for this DTDS, the positive equilibrium (fixed point) is STABLE.

∴ In the long term, the equilibrium harvest will be $\boxed{Y(h) = hx^* = h(0.6 - 0.4h)}$

(What we harvest in long term depends on our harvesting intensity parameter h)

(b) What is the optimal harvesting intensity for this crop? What is the maximum yield's value?

Now, we want the GLOBAL MAX of our long term yield $Y(h)$ for $h \in (0, 1.5)$

Note: this is the range for h where
1. $h > 0$ so we actually harvest some apples each week
2. $h < 1.5$ so the equilibrium $x^* = 0.6 - 0.4h$ is > 0 (which means quantity of apples is > 0 in long term)

which means we weren't so greedy as to harvest our orchard to death!

$Y(h) = 0.6h - 0.4h^2$

$\Rightarrow Y'(h) = 0.6 - 0.8h$ (same domain as $Y \Rightarrow$ no type 2 crit. #s)

type 1? $0 = Y'(h)$

$0 = 0.6 - 0.8h$

$0.8h = 0.6$

$h = \frac{0.6}{0.8} = 0.75$

$\therefore Y(h)$ has one critical # $h = 0.75$

but is it a GLOBAL MAX?

broken domain (intervals)	$(0, 0.75)$	$(0.75, 1.5)$
sign of $Y'(h)$	$Y'(0.5) > 0$ ⊕	$Y'(1) < 0$ ⊖
behaviour	INCREASING	DECREASING

Y has a LOCAL MAX @ $h = 0.75$

For this function, it is in fact a GLOBAL MAX since the function increases from $0 < h < 0.75$ then decreases for $h > 0.75$. The function's behaviour does not allow the possibility to reach a bigger value than $Y(0.75) = 0.225$

$\therefore$ the GLOBAL MAX long-term yield occurs when $h = 0.75$ and the maximum long-term yield is $Y(0.75) = 0.225$ (in 100 000's of kgs of apples)

Example 13.4. The yield of another crop changes with the amount of fertilizer (such as Nitrogen). When nitrogen levels in the soil are low, then adding some nitrogen will greatly increase the yield. When nitrogen levels are already very high, then adding some nitrogen might actually decrease the crop's yield. Assume that the crop's yield as a function of the amount of nitrogen in the soil is given by

$$Y(N) = \frac{N}{1 + N^2} \qquad \text{(where the nitrogen level } N \text{ is measured in } \mu\text{g nitrogen/kg soil)}$$

What is the optimal level of nitrogen in the soil?

Now, we want the GLOBAL MAX of the crop's yield function $Y(N)$ for $N \geq 0$ (Nitrogen level cannot be negative.)

$Y'(N) = \frac{(1)(1+N^2) - N(2N)}{(1+N^2)^2}$ (quotient rule)

$\Rightarrow Y'(N) = \frac{1 - N^2}{(1+N^2)^2}$ ← same domain as Y so no type 2 crit. #s

type 1? $0 = Y'(N)$

$\Rightarrow 0 = \frac{1 - N^2}{(1+N^2)^2}$

$\Rightarrow 0 = 1 - N^2$

$\Rightarrow 0 = (1 - N)(1 + N)$

$\therefore Y(N)$ has one critical # $N = 1$ ~~$N = -1$~~ ← not relevant for nitrogen level.

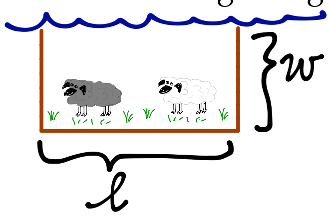
broken domain (intervals)	$[0, 1)$	$(1, \infty)$
sign of $Y'(N)$ on interval	$Y'(0.5) > 0$ ⊕	$Y'(3) < 0$ ⊖
behaviour of Y on interval	INCREASING	DECREASING

$\therefore Y(N)$ has a LOCAL MAX at $N = 1$

It is also a GLOBAL MAX since Yield increases for $0 \leq N < 1$, then decreases for all $N > 1$

$\therefore$ the maximum yield is attained when Nitrogen levels are at 1 μg/kg soil

Example 13.5. Near her crops, a farmer wants to enclose a rectangular grazing area for sheep alongside a (very straight) river. She has 1800 feet of fencing to use. What dimensions will provide the maximum grazing area for the sheep?



A = area enclosed
ℓ = length of side parallel to waterline
w = width of area

we want the absolute MAX area subject to $\begin{cases} 0 < w < 1800 \\ 0 < \ell < 1800 \\ \ell + 2w = 1800 \end{cases}$

because farmer only has 1800 feet of fencing.

We know $A = \ell \cdot w$ but this is a function of 2 variables.

Since total amount of fencing needed is $w + \ell + w = \ell + 2w = 1800$, we can isolate (either) one of the variables.

Let's isolate ℓ: $\quad \ell + 2w = 1800$

$\Rightarrow \ell = 1800 - 2w$

$\therefore A = (1800 - 2w) \cdot w$

Now $A(w)$ is a function of one variable so we know how to find its extreme values.

$A(w) = 1800w - 2w^2$

$A'(w) = 1800 - 4w$

$0 = A'(w)$

$0 = 1800 - 4w$

$\Rightarrow w = 450$

crit.# $w = 450$

broken domain	$(0, 450)$	$(450, 1800)$
sign of $A'(w)$ on interval	$A'(1) > 0$ ⊕	$A'(1000) < 0$ ⊖
behaviour	INCREASING ↗	DECREASING ↘

Since $A(w)$ increases for $0 < w < 450$, then decreases for all $w > 450$, the area attains a GLOBAL MAX when $w = 450$ $\therefore \ell = 1800 - 2(450) = 900$

So the absolute MAX area is $A(450) = 405\,000 \text{ ft}^2$ when $w = 450$ ft and $\ell = 900$ ft

Study Guide

- ◇ extreme values: **absolute/global min/max** **relative/local min/max values**
- ◇ **Extreme Value Theorem**: how to find absolute extrema of a continuous function over a closed interval.
- ◇ **Optimization**: convert optimization into a calculus problem about extreme values

Exercises Adler & Lovrić textbook, 2nd ed.	§6.1 pg. 363 # 7,9, 13, 15, 17, 19, 21–43, 63, 64, 81 §6.2 pg. 375 #17-18, 21-22
Prof. Lutscher's Notes (on Virtual Campus)	§12.1 pg. 45 # 1–4 §13.1 pg. 53 # 1, 4, 5, 7, 9, 10