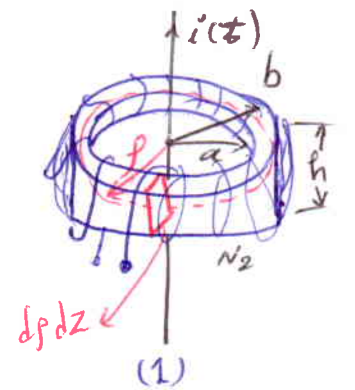
$M = k\sqrt{L_{11}L_{22}}$, $k = \sqrt{k_1 k_2}$ $\Rightarrow$ $\boxed{M = \sqrt{k_1 k_2 L_{11} L_{22}}}$

$k =$ coefficient of coupling between 2 coils. $0 \le k \le 1$

if $k = 1 \Rightarrow$ coils are "Perfectly coupled".

$\mathcal{R}$ = reluctance of the circuit $\Rightarrow$ $\boxed{M = k \dfrac{N_1 N_2}{\mathcal{R}}}$ To calculate mutual inductance between any two coils for a linear m - circuit.

EX: A toroidal coil of 2000 turns is wound over a m-ring with inner radius of $a = 10$ mm , outer $b = 15$ mm , height $h = 10$ mm, μ_r = relative permeability $= 500$. A very long wire passing the center of coil carries a time-varying current. Determine the mutual inductance between the toroid & the straight conductor:



Ampère's law for m-flux density at any radius ρ within the toroid: $\vec{B}_1 = \dfrac{\mu i}{2\pi\rho}\hat{\varphi}$

The flux linking the toroid $\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2 = \dfrac{\mu i}{2\pi}\int_a^b \dfrac{d\rho}{\rho}\int_0^h dz$

$$= \frac{\mu i}{2\pi}\ln(b/a)h$$

$$\rightarrow L_{21} = N_2 \frac{d\Phi_{21}}{di} = \frac{N_2 \mu h}{2\pi}\ln(b/a) \quad ; \mu = \mu_0 \mu_r$$

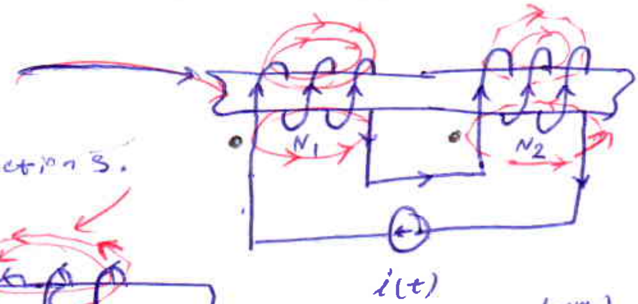
$$= \frac{(2000)[(500)(4\pi\times10^{-7})](0.01)}{2\pi}\ln(15/10) = 0.81 \text{ mH}$$

*

Inductance of coupled coils:

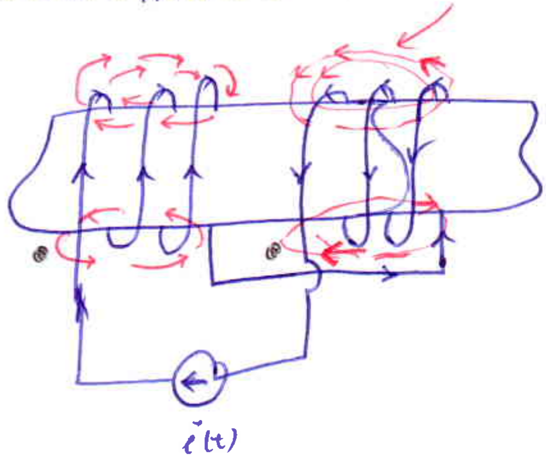
series connection of two coils: connected in tandem (end to end):

- series-aiding: flux in the same direction.
- series-opposing: produce fluxes in opposite directions.



both currents enter the (leave) dotted terminal → aid

→ One enter one leave → oppose.

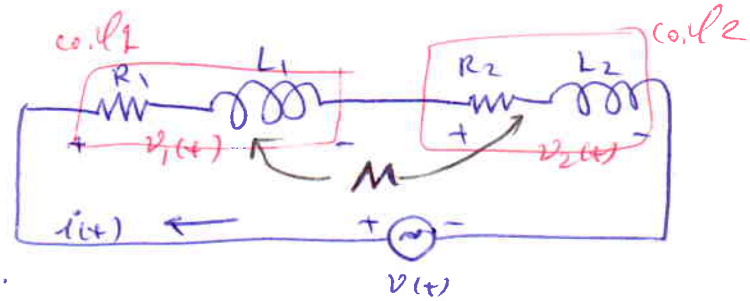


↓

if the flux of one coil is normal to the flux of second coil, the two coils are independent ⇒ the mutual inductance between them is Zero.

↳ the m-axes of the two coils are normal to each other.

→ $\begin{cases} L_1 = \text{inductance of coil 1.} \\ L_2 = \text{- - - coil 2.} \end{cases}$



M = mutual inductance of two coils.

→ R_1, R_2: internal resistances.

Voltage drop across coil 1 $\begin{cases} v_1 = L_1 \dfrac{di}{dt} + iR_1 \pm M\dfrac{di}{dt} \\ v_2 = L_2 \dfrac{di}{dt} + iR_2 \pm M\dfrac{di}{dt} \end{cases}$

$\begin{cases} + \to \text{series-aiding} \\ - \to \text{" -opposing} \end{cases}$

Kirchhoff's voltage law: $v = (L_1 + L_2 \pm 2M)\dfrac{di}{dt} + i(R_1 + R_2) \equiv L\dfrac{di}{dt} + Ri$

effective inductance

eff. resistance

$$\begin{cases} L = L_1 + L_2 \pm 2M \\ R = R_1 + R_2 \end{cases}$$

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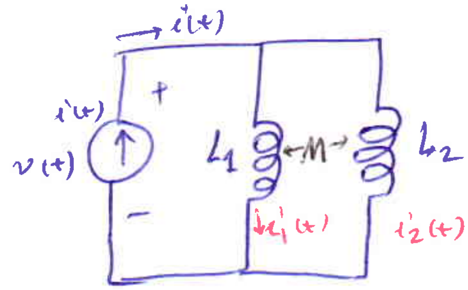
Parallel connection of two coils:



$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$$

$\rightarrow$ $-$: parallel-aiding; $+$: - - - oppose

Ex: The self-inductances of 2 coils are 800 mH & 200 mH. The coefficient of coupling is 0.8. effective inductance when connected in a) parallel-aid b) parallel-opposing.

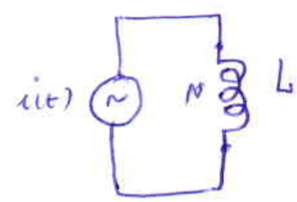
$$\begin{cases} L_1 = 0.8 \text{ H} \\ L_2 = 0.2 \text{ H} \\ k = 0.8 \end{cases} \Rightarrow M = k\sqrt{L_1 L_2} = 0.8\sqrt{(0.8)(0.2)} = 0.32 \text{ H}$$

$$a)\ L = \frac{(0.8)(0.2) - 0.32^2}{0.8 + 0.2 - 2(0.32)} = 0.16 \text{ H} = 160 \text{ mH}$$

$$b)\ L = \frac{(0.8)(0.2) - 0.32^2}{(0.8) + 0.2 + 2(0.32)} = 0.035 \text{ H} = 35 \text{ mH}$$

Energy in m-field stablished by coils:

- single coil: when i(t) is increasing, the induced emf across its terminals: $e = -N\frac{d\phi}{dt}$



Work done by the source to maintain an increase in the current in dt:

$$dw = \underbrace{-e}_{v}\,\underbrace{i\,dt}_{q} = i N d\Phi$$

source supplys energy; coil absorbs energy.

$$\Rightarrow W = N\int i\, d\phi$$

$\rightarrow$ How Φ varis with current i; For Linear m-circuit: $N d\phi = L di \rightarrow dw = L i di$

$$\int_{w_0}^{w_f} dw = \int_{I_0}^{I_f} L i\, di \Rightarrow w = w_f - w_0 = \frac{1}{2}LI_f^2 - \frac{1}{2}LI_0^2$$

If $I_0 = 0 \Rightarrow$ $\boxed{W = \text{energy stored in m-circuit} = \frac{1}{2} L i^2}$

For Linear m-circuit:

if $\lambda = N\phi$ = total flux Linkages with the coil $\Rightarrow W = \frac{1}{2} N\phi i = \frac{1}{2} \lambda i$.

$$\left\{ \begin{array}{l} \Phi = \int_s \vec{B} \cdot d\vec{S} = BA \\ Ni = \oint_c \vec{H} \cdot d\vec{\ell} = H\ell \end{array} \right.$$

A → cross-sectional area of coil

ℓ → length of coil

Ni → Total current enclosed by contour

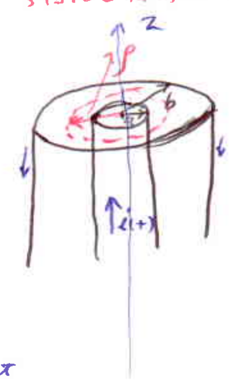
$\Rightarrow$; $A\ell$ = volume enclosed by coil.

$\rightarrow W = \frac{1}{2} N\phi i = \frac{1}{2} BHA\ell = \frac{1}{2} BH(\text{Volume})$

$\Rightarrow \boxed{w_m = \frac{1}{2} BH = \frac{1}{2}\mu H^2 = \frac{1}{2\mu} B^2 = \frac{1}{2} \vec{B} \cdot \vec{H}}$

energy stored ~~is distributed~~ throughout its m-field region. [common: energy is stored in the inductor].

EX: $\left\{ \begin{array}{l} a = \text{outer radius of inner conductor} \\ b = \text{inner radius of outer conductor} \end{array} \right.$, coaxial transmission line,



determine the inductance per unit length of the line using energy concept:

$a \le \rho \le b$: $H = \frac{i}{2\pi\rho} \Rightarrow w_m = \frac{1}{2}\mu \left[\frac{i}{2\pi\rho}\right]^2$ energy density at any point within the region.

$$\Rightarrow W_m = \int w_m \, dv = \frac{i^2 \mu}{8\pi^2} \int_a^b \frac{\rho}{\rho^2} d\rho \int_0^{2\pi} d\varphi = \frac{\mu}{4\pi} i^2 \ln\left(\frac{b}{a}\right) \quad [J/m].$$

$$W = \frac{1}{2} L i^2 \Rightarrow \boxed{L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) \quad [H/m]}$$

Coupled coils:

Φ_1 = total flux linking coil 1 carrying i_1 $= \Phi_{11} + \Phi_{12}$

Φ_{11} → flux by 1 when $i_2 = 0$

Φ_{12} → flux by i_2 in coil 2 and linking coil 1 when $i_1 = 0$.

$\Phi_2 = \Phi_{22} + \Phi_{21}$

$\Rightarrow W$ = magnetic energy stored in the region $= \frac{1}{2} N_1 \Phi_1 i_1 + \frac{1}{2} N_2 \Phi_2 i_2$

$= (\frac{1}{2} N_1 \Phi_{11} i_1 + \frac{1}{2} N_1 \Phi_{12} i_1) + (\frac{1}{2} N_2 \Phi_{22} i_2 + \frac{1}{2} N_2 \Phi_{21} i_2)$

$L_{11} = \frac{N_1 \Phi_{11}}{i_1}$, $L_{22} = \frac{N_2 \Phi_{22}}{i_2}$

$L_{12} = \frac{N_1 \Phi_{12}}{i_2}$, $L_{21} = \frac{N_2 \Phi_{21}}{i_1}$ $\Rightarrow$ $W = \frac{1}{2} L_{11} i_1^2 + \frac{1}{2} L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2 + \frac{1}{2} L_{21} i_1 i_2$

if $M = L_{12} = L_{21}$ for linear m-circuit

fluxes aid each other. $\leftarrow$ $W = \frac{1}{2}[L_{11} i_1^2 + L_{22} i_2^2 + 2M i_1 i_2]$

fluxes opposes each other $\leftarrow$ $W = \frac{1}{2}[L_{11} i_1^2 + L_{22} i_2^2 - 2M i_1 i_2]$

Two coils in series: $i_1 = i_2 = i \rightarrow$ aiding: $W = \frac{1}{2}[L_{11} + L_{22} + 2M] i^2$

opposing: $W = \frac{1}{2}[L_{11} + L_{22} - 2M] i^2$

$\downarrow$

equivalent inductance of the coupled circuit.

EX: Two coils are connected in series-aiding and the current goes from 2A to 5A.

a) Initial energy? b) final energy? c) the change in the stored energy?

$L_{eff\text{-}aid} = 2.38$ H , $L_{eff\text{-}oppos} = 1.02$ H

a) $W_i = \frac{1}{2} L_{eff} i_o^2 = \frac{1}{2}(2.38)(2^2) = 4.76$ J

b) $W_f = \frac{1}{2} L_{eff} i_f^2 = \frac{1}{2}(2.38)(5^2) = 29.75$ J

c) $\Delta W = W_f - W_i = 29.75 - 4.76 = 24.99$ J.

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Maxwell's eqn from Ampère's law:

$$\boxed{\oint_C \vec{H}\cdot d\vec{l} = I.}$$

$$\Downarrow$$

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

- H = m-field intensity
- I = uniform current enclosed by countour $\underline{c}$.
- $\vec{J}$ = vol. current density over surface $\underline{S}$ bounded by closed $\underline{c}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J}$$

$$\boxed{0 = \vec{\nabla}\cdot\vec{J}}$$

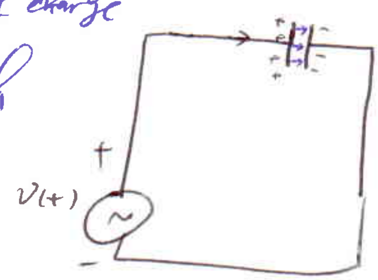
but for time varying fields we had:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$\vec{J}$ is not continuous in general (non diverging). $\rightarrow$ so $\vec{\nabla}\times\vec{H} = \vec{J}$ leads to a contradictions in time-varying case.

A capacitor connected to a time-varying voltage source:

rise & fall of applied voltage in time $\rightarrow$ amount of charge that is transferred from the source to each electrode.



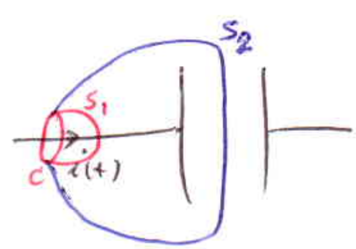
$\rightarrow$ charging a capacitor is time-varying process.

$\rightarrow$ there is a time-varying current in the circuit $i(t)$. $\rightarrow$ this current establishes a time-varying m-field in the region.

$\rightarrow$ select an open surface $\underline{S}$ bounded by a closed contour $\underline{c}$:

Ampère's law: $\oint_C \vec{H}\cdot d\vec{l} \overset{S_1}{=} i$

$$\oint_C \vec{H}\cdot d\vec{l} \overset{S_2}{=} 0$$



Maxwell Prediction: there must exist a current in the capacitor.

⇒ Maxwell said it is a conduction current: "displacement current".

→ add another term to the Ampère's law for conservation of charge:

$\vec{\nabla}\cdot\vec{D} = \rho_v$ Gauss's law and

$\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$ eqn. of continuity.

time & space are indep. variable → change the order of differentiation:

$$\vec{\nabla}\cdot\vec{J} = -\vec{\nabla}\cdot\left(\frac{\partial\vec{D}}{\partial t}\right) \Rightarrow \vec{\nabla}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) = 0 \rightarrow$$

is a continuous field.

$$\rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}}$$

displacement current. density (A/m^2).

total current density at any point in a medium $= \vec{J} + \frac{\partial\vec{D}}{\partial t}$. → unification of electromagnetic field theory.

maxwell predicted that Em-fields must propagate through space as wave (speed of light).

Hertz: experimentally showed that this current do exist.

$$\boxed{\oint_C \vec{H}\cdot d\vec{l} = \underbrace{\int_S \vec{J}\cdot d\vec{S}}_{\text{conduction current}} + \underbrace{\int_S \frac{\partial\vec{D}}{\partial t}\cdot d\vec{S}}_{\text{displacement current}}}$$

→ communications.

EX! $\vec{H} = H_0 \sin\theta\, \hat{y}$ A/m, $\theta = \omega t - \beta z$, β = const. a) displacement current density

b) e-field intensity. (in free space).

$\vec{J} = 0$ in free space. ; $\frac{\partial\vec{D}}{\partial t} = \vec{\nabla}\times\vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_0\sin\theta & 0 \end{vmatrix} = -\frac{\partial}{\partial z}[H_0\sin\theta]\,\hat{x} + \frac{\partial}{\partial x}[H_0\sin\theta]\,\hat{z}$

$= \beta H_0 \cos\theta\, \hat{x}$ A/m^2

$$\frac{\partial \vec{D}}{\partial t} = \beta H_0 \cos\theta\,\hat{x} \rightarrow \vec{D} = \frac{\beta}{\omega} H_0 \sin\theta\,\hat{x} \quad C/m^2$$

$$\rightarrow \vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{\beta}{\omega \epsilon_0} H_0 \sin\theta\,\hat{x} \quad V/m.$$

---

maxwell's eqns from Gauss's law : $\vec{\nabla}\cdot\vec{D} = \rho_v$ → free vol. charge density in the medium

$\rho_v = 0$ within a dielectric medium

because the effect of polarized charges is already included in ϵ_r.
dielectric const.

⇒ for time-varying case, all arguments are the same, but $\vec{D}$ & ρ_v are time-dep. field quantities. ⇒ $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_v}$ Maxwell's eqn.

$$\oint_s \vec{D}\cdot d\vec{S} = \int_v \rho_v\,dv = q(t).$$

$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ → Gauss's law for magnetostatic fields is Maxwell's eqn.

$\vec{B} = \vec{B}(t)$

$$\boxed{\oint \vec{B}\cdot d\vec{S} = 0}$$

Maxwell's eqns are complete.

Maxwell's eqns & boundary conditions :

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} \quad (1)$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S} \quad (2)$$

$$\vec{\nabla}\cdot\vec{D} = \rho_v \Rightarrow \oint_S \vec{D}\cdot d\vec{S} = \int_v \rho_v\,dv \quad (3)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \oint_S \vec{B}\cdot d\vec{S} = 0 \quad (4)$$

$\vec{E}$ = e-field intensity V/m.
$\vec{H}$ = m-field 〃 A/m.

$\vec{D}$ = e-flux density C/m^2.
$\vec{B}$ = m-flux 〃 Wb/m^2 (T).

ρ_v = free volume charge density C/m^3.

$$\begin{cases} I = \int_s \vec{J} \cdot d\vec{s} \\ q = \int_v \rho_v \, dv \end{cases}$$

$\begin{cases} I = \text{e-current through surface } \underline{s}, \underline{A} \\ q = \text{free charge enclosed in volume } v, \underline{C} \end{cases}$

(1): A time-varying m-field produces a time-varying e-field : transformers & induction motors operate on this principle.

(2) A time-varying m-field can be produced by conduction current & by displacement current.

$\frac{\partial \vec{D}}{\partial t}$ = the rate of change of e-flux density, suggest

∴ A time-varing e-field creates a time-varying m-field, and then time-varying m-field produces time-varying e-field.

→ energy from e-field can be transferred to the m-field, which then transfers to e-field. → continuation of transfer of energy from one field to another, maxwell predicted the propagation of em-energy in any medium.

→ em-fields travel as waves helped maxwell to predict velocities & other peculiar characteristics of these waves. → velocity of these waves in free space is equall to $\underline{c}$ ⇒ light has the same nature of em-waves.

→ 1880s, Hertz showed experimentally the existence of em-waves → confirm maxwell's predictions.

(3) Total e-flux emanating from a closed volume at any time = charge enclosed.

If $\text{charge}_{en.} = 0$ → e-flux lines are continuous.

(4) m-flux lines are always continuous : the net m-flux emerging from any closed surface at any time = 0.

continuity eqn: $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t} \Rightarrow \oint_S \vec{J}\cdot d\vec{s} = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$

Lorentz force eqn: $\vec{F} = q(\vec{E} + \vec{u}\times\vec{B})$ [N]. ; charge q is moving with $\vec{u}$ in em-fields.

force acting on charge & current

per unit volume: $\vec{f} = \rho_v\vec{E} + \vec{J}\times\vec{B}$ $[N/m^3]$

; $\vec{J} = \rho_v\vec{u}$ $[A/m^2]$.

* maxwell's eqns with continuity & Lorentz force: completely describe the interactions among charges, currents, and e- & m-fields in any medium.

constitutive eqns: In a linear, homogeneous & isotropic medium:

$\vec{D} = \epsilon\vec{E}$; ϵ = permittivity F/m

$\vec{J} = \sigma\vec{E}$; conductivity S/m → ohm's law: motion of a charge in a conductor by e-field produces a current in the conductor.

$B = \mu\vec{H}$; permeability H/m

Boundary conditions: em-fields obtained from the solution of maxwell's eqns must satisfy the boundary conditions at interface between different media.

B.C. for time-varying fields are exactly the same as those for static fields.

$$\begin{cases} E_{t1} = E_{t2} \\ H_{t1} - H_{t2} = J_s \end{cases} \qquad \begin{cases} \hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0 \\ \hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{J}_s \end{cases}$$

$$\begin{cases} B_{n1} = B_{n2} \\ D_{n1} - D_{n2} = \rho_s \end{cases} \qquad \begin{cases} \hat{n}\cdot(\vec{B}_1 - \vec{B}_2) = 0 \\ \hat{n}\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s \end{cases}$$

$$\begin{cases} J_{n1} = J_{n2} \\ \frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2} \end{cases} \qquad \begin{cases} \hat{n}\cdot(\vec{J}_1 - \vec{J}_2) = 0 \\ \hat{n}\times\left[\frac{\vec{J}_1}{\sigma_1} - \frac{\vec{J}_2}{\sigma_2}\right] = 0 \end{cases}$$