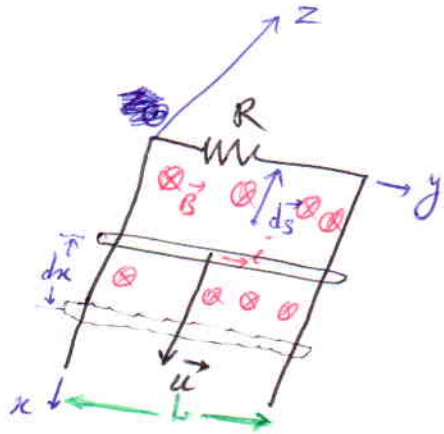
Faraday's law of induction:



$d\vec{s} = L\,dx\,\hat{z}$ → cross-sectional area of the closed circuit.

$d\phi = \vec{B}\cdot d\vec{s} = -BL\,dx$ → m-flux passing through the plane of closed loop.

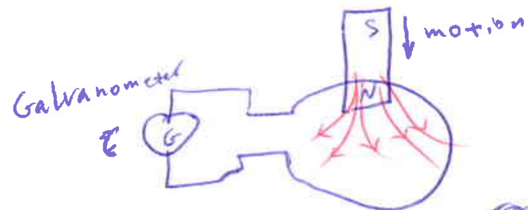
$\frac{d\phi}{dt} = -BL\frac{dx}{dt} = -BLu$ → $\boxed{e = -\frac{d\phi}{dt}}$ Faraday's law.

e: induced emf around a closed path

$\frac{d\phi}{dt}$: rate of change of the m-flux w.r. time passing through the area enclosed by the path.

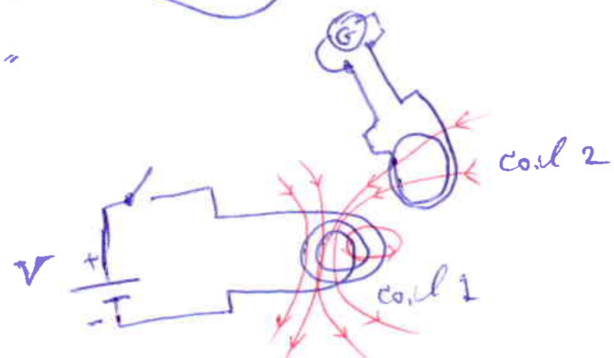
$-$: Lenz's law

Faraday: with experiments with stationary coils, he discovered that an emf is induced in a coil when a time-varying m-flux passes through the area enclosed by the coil.



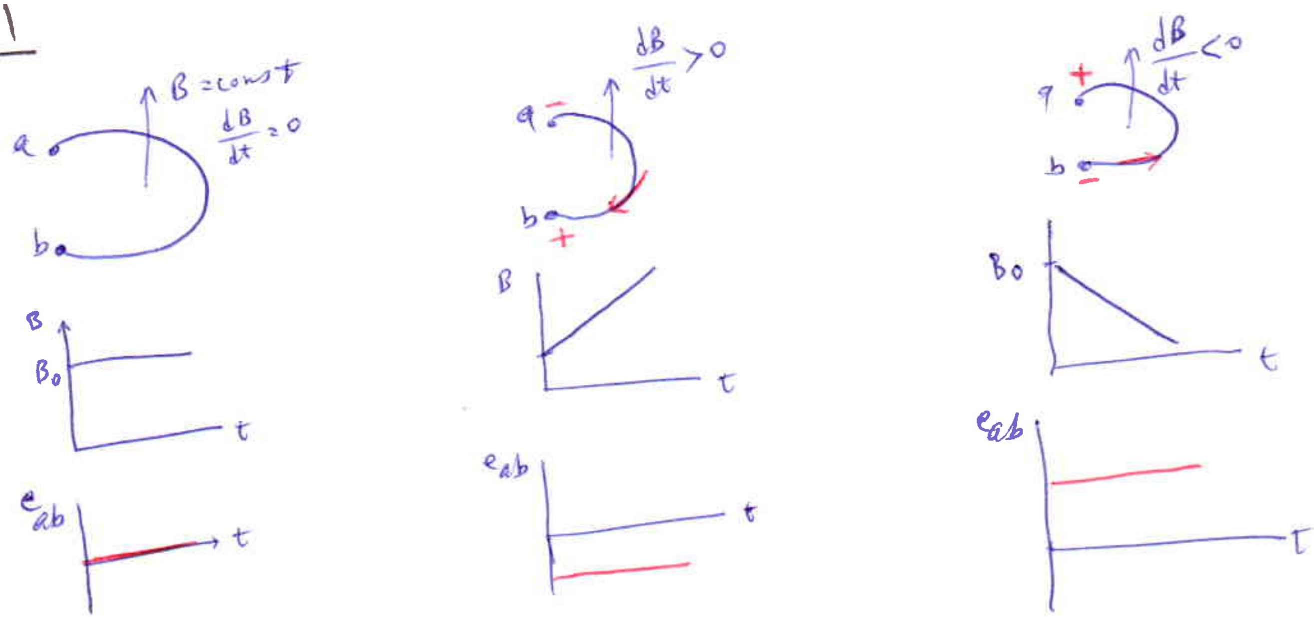
$\mathcal{E}_{ind} \rightarrow i_{ind}$

As far as emf_{ind} is concerned, ⇐ "Em-induction"
the closed path does not have to be conductive.
It can happen in free space or insulating medium.



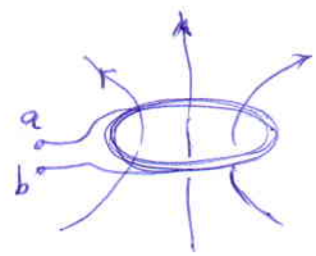
Lenz's law: The induced current tends to counterbalance the change in the original m-flux. (conservation of energy).

- direction of current in a closed loop
- polarity of the induced emf in an open loop (look if it was closed)



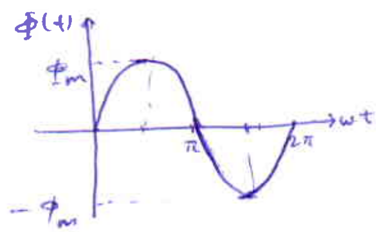
Induced emf: $\boxed{e = -N\frac{d\Phi}{dt} \qquad e = -\frac{d\lambda}{dt}}$; Flux linkage : $\lambda = N\Phi$

sinusoidally varying the flux linkage:



$$\Phi = \Phi_m \sin\omega t$$

$$e_{ab} = -N\frac{d\phi}{dt} = \underbrace{-N\omega\Phi_m}_{\mathcal{E}_m}\cos\omega t$$



freq.

$$\mathcal{E}_{\substack{effective\\(rms)}} = \frac{1}{\sqrt{2}}\mathcal{E}_m = \underbrace{\sqrt{2}\,\pi}_{\simeq 4.44} f N \Phi_m$$

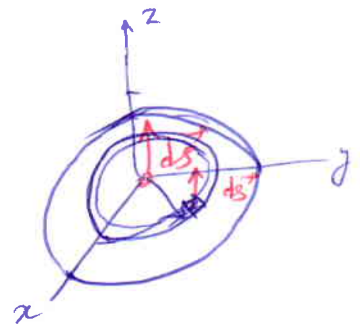
$$\boxed{\mathcal{E}_{eff.} = 4.44 fN\Phi_m}$$ → operating principle of a transformer.

Transformer equation

"transformer emf".

112

EX: A circular conducting loop of radius 40 cm lies in xy-plane, has resistance of 20 Ω. $\vec{B}^{[T]} = 0.2\cos 500t\,\hat{x} + 0.75\sin 400t\,\hat{y} + 1.2\cos 314t\,\hat{z}$ determine the effective value of the induced current in the loop.



$$d\vec{S} = \rho(\rho\, d\phi)\,\hat{z}$$

$$d\Phi = \vec{B}\cdot d\vec{S} = 1.2\,\rho\, d\rho\, d\phi \cos 314t$$

$$\Phi = \int d\phi = 1.2\,\rho\cos 314t \int_{\rho=0}^{0.4} \rho\, d\rho \int_{\varphi=0}^{2\pi} d\varphi$$

$\Phi = 0.603 \cos 314t$ wb, total flux linking the loop at any time t.

$$\omega = 314 \,\text{rad/s} = 2\pi f \Rightarrow f = 50^{HZ}.$$

$$\Phi_{max} = 0.603 \Rightarrow \mathcal{E}_{eff} = (4.44)(50)(1)(0.603) = 133.866^{V}.$$

$$I_{eff} = \frac{\mathcal{E}_{eff}}{R} = \frac{133.866}{20} = 6.693^{A}.$$

Maxwell's eqn (Faraday's law):

→ To sustain a current within the conductor, there must exist an e-field inside a conductor.

$e = \oint_C \vec{E}\cdot d\vec{l}$ (induced emf) ; C → direction of i_{ind}.

$$; \Phi = \int_S \vec{B}\cdot d\vec{S} \rightarrow \boxed{\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}}$$

$d\vec{s}$: surface bounded by c, right hand rule.

integral form of Faraday's law.

If surface to be fixed in space $\frac{\partial S}{\partial t} = 0 \Rightarrow$ $\oint_C \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$ maxwell's eqn. $\int$ form of transformer eqn.

since $\oint_C \vec{E}_{ind}\cdot d\vec{l} \neq 0$ → Induced e-field is not conservative.

differential form of Faraday's law.

Stoke's theorem: $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial B}{\partial t}\cdot d\vec{s}$ → $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

maxwell's eqn.

→ using this eqn, we can compute the e-field intensity at a fixed point in space when the B-field is a function of time.

$$\boxed{e_t = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}}$$

"transformer emf"

General form of maxwell's eqn:

The motion of a loop in a m-field produces emf (e_m):

$$e_m = \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$$

For both motion & time-varying B-field:

$$\Rightarrow e = e_t + e_m = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}$$

most general form of maxwell's eq. for Faraday's law. (S)

$$\rightarrow \boxed{\oint \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_C (\vec{u}\times\vec{B})\cdot d\vec{l}}$$

→ Stoke's theorem

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \int_S [\vec{\nabla}\times(\vec{u}\times\vec{B})]\cdot d\vec{s}$$

$$\rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times(\vec{u}\times\vec{B})}$$

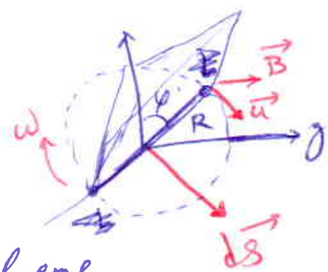
most general (Faraday) (Point form).

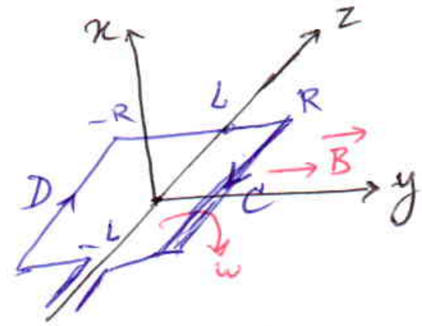
↓

calculate e-field at a point of observation moving with a velocity $\vec{u}$ in a m-field $\vec{B}$.

EX: A rectangular coil of N-turns is rotating in a uniform m-field. Find induced emf in the coil using a) The concept of motional emf, b) Faraday's law of induction.

a) B = uniform; emf only by the motion of loop.





N conductors at C: $\vec{u} = \omega R\,\hat{\varphi}$

$$\begin{cases} \varphi = \omega t \\ \vec{B} = B\hat{y} \end{cases} \rightarrow \vec{u}\times\vec{B} = -\omega R B \sin\omega t\,\hat{z}$$

$A = 4LR$ area of loop.

$$e_m = \int_C N(\vec{u}\times\vec{B})\cdot d\vec{\ell}$$ → motional emf in N conductors at C.

$$= -NBR\omega \sin\omega t \int_L^{-L} dz = 2NLRB\omega \sin\omega t \Rightarrow e = 2e_m = 4LRNB\omega \sin\omega t$$

$$\boxed{e = NBA\omega \sin\omega t}$$

b) Faraday's law: $\Phi = \int_S \vec{B}\cdot d\vec{S} = \int_S (\hat{y}\cdot\hat{\rho})B\,dS = B\cos\omega t \int_S dS = BA\cos\omega t$

$$\rightarrow \boxed{e = -N\frac{d\Phi}{dt} = NBA\omega \sin\omega t}$$

EX: Now we add changing in $\vec{B} = B_m \sin\omega t$, determine the induced emf using (a) motional & transformer emfs and (b) Faraday's law of induction.

$$a)\ e_{m_{tot.}} = 2e_m = 2N\int_C (\vec{u}\times\vec{B})\cdot d\vec{\ell} = 2N\int_C (\omega R\hat{\varphi} \times B_m \sin\omega t\,\hat{y})\cdot d\vec{\ell}$$

$$= 2N\omega R B_m \sin^2\omega t \int_L^{-L} dz = B_m A N\omega \sin^2\omega t \quad ;\ A = 4LR$$

transformer emf

$$e_t = -N\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -N\omega B_m \cos\omega t\,(\hat{y}\cdot\hat{\rho})\int_S dS = -B_m A N\omega \cos^2\omega t$$

$$\Rightarrow e = e_{m_{tot}} + e_t = -B_m A N\omega(\cos^2\omega t - \sin^2\omega t) = -B_m A N\omega \cos 2\omega t$$

b) Faraday's law. $\Phi = \int_S \vec{B} \cdot d\vec{S} = B_m \sin\omega t \cos\omega t \int_S dS = \frac{1}{2} B_m A \sin 2\omega t$

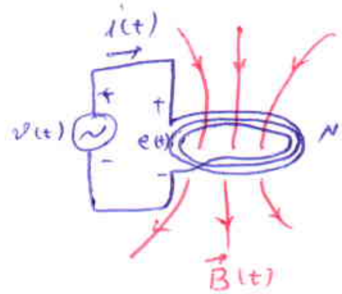
$$\Rightarrow e = -N\frac{d\phi}{dt} = -\frac{1}{2} B_m AN \frac{d}{dt}(\sin 2\omega t) = -B_m A N \omega \cos 2\omega t \checkmark$$

---

self-inductance :

N-turn closely wound coil connected to a time-varying source, carrying $i(t)$:

$i(t) \rightarrow$ generate a t-varying flux $\Phi(t)$ $\longrightarrow$ emf in the coil.



induced emf causes an induced current in the coil which tends to oppose the change in the original current $i(t)$.

$\rightarrow \Phi(t)$: flux at any instant linking all the turns in the coil $\rightarrow$

$\rightarrow$ induced emf $e(t) = N\frac{d\Phi}{dt}$ $\rightarrow$ opposes the applied voltage (induced voltage, back emf, or counter emf).

(−) sign in the figure.

applied voltage

$$v = e = N\frac{d\Phi}{dt}$$

$\lambda = N\Phi \rightarrow$ number of flux linkage.

$$L = \frac{d\lambda}{di} = N\frac{d\Phi}{dt}$$

self inductance (inductance of the coil): The rate of change of flux linkages per unit change in the current.

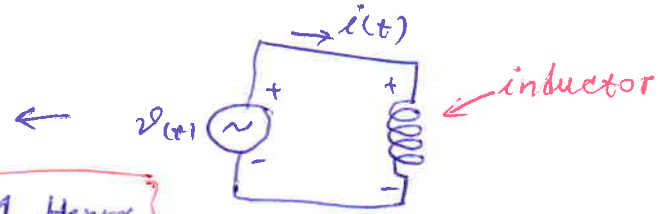
$\left[\frac{Wb.turn}{Ampere}\right]$ = Henry.

$\Rightarrow \underbrace{L\,di}_{=d\lambda = v} = N\,d\phi$ $\longrightarrow$ If the material that coil is wound is a linear (const. Permeability) m-material, L is constant. $\Rightarrow \phi(t) \propto i(t)$

$Li = N\phi \rightarrow \boxed{L = \frac{N\phi}{i}}$ inductance of a m-circuit for a steady current in the coil.

$$\Rightarrow L\frac{di}{dt} = N\frac{d\phi}{dt} \Rightarrow \boxed{v = L\frac{di}{dt}}$$ circuit equation to find voltage drop across an inductance L.

If the voltage drop across an element is one volt, and the current on that element changes 1 A/sec $\Rightarrow \boxed{L = 1 \text{ Henry.}}$



$\underline{L}$ inductance only depends upon the parameters of the m-circuit.

$$\Rightarrow L = \frac{N^2\phi}{Ni} = \frac{N^2\phi}{\mathcal{F}} = \frac{N^2}{\mathcal{R}} = \mathcal{P}N^2$$

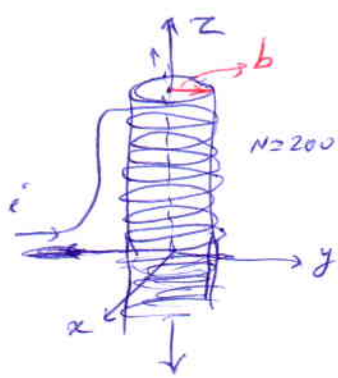
$\mathcal{F} = Ni$ applied mmf to m-circuit

reluctance $\mathcal{R} = \mathcal{F}/\phi$, $\mathcal{P} = \frac{1}{\mathcal{R}}$

Permeance of the m-circuit.

$\Rightarrow$ each of m-circuits can now be represented by an equivalent e-circuit in terms of its inductance.

EX: A very long cylinder of radius 20 cm is closely & tightly wound with 200 turns per unit length to form an air-core inductor (solenoid). If the current is constant, determine its inductance.



$\vec{B} = \mu_0 n I \hat{z}$; n = # of turns per unit length.

The flux enclosed by a cylinder of radius b:

$$\Phi = \int_s \vec{B}\cdot d\vec{S} = \int_s \mu_0 n I \hat{z}\cdot \rho\, d\rho\, d\phi\, \hat{z} = \mu_0 n I \int_0^b \rho\, d\rho \int_0^{2\pi} d\phi$$

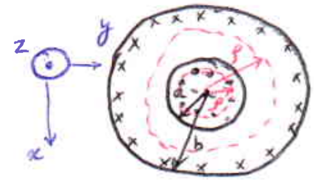
$$= \mu_0 n I \pi b^2 \Rightarrow L = \frac{N\Phi}{i} = \frac{n\ell\Phi}{i} \Rightarrow \frac{L}{\ell} = \frac{n\Phi}{i} = \mu_0 \pi n^2 b^2$$

ℓ: length of solenoid

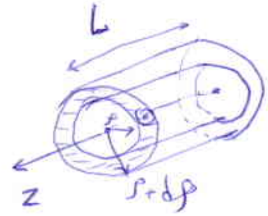
$$\Rightarrow L = (4\pi \times 10^{-7})\,\pi\,(200)^2\,(0.2^2) = 6.32 \text{ mH/m}$$

EX : Find self-inductance per unit length of a coaxial cable of inner conductor $\underline{a}$ & outer $\underline{b}$. The outer conductor has negligible thickness. The current is uniformly distributed inside the inner conductor.

$\vec{J} = \frac{I}{\pi a^2}\hat{z}$ $\qquad B(2\pi\rho) = \mu_0 \int \vec{J}\cdot d\vec{s} = \mu_0 \frac{I}{\pi a^2}\pi\rho^2 = \mu_0 \frac{I\rho^2}{a^2}$



Apmère's law for $0 \le \rho \le a$: $\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I \Rightarrow \boxed{\vec{B}_i = \frac{\mu_0 I \rho}{2\pi a^2}\hat{\phi}}$



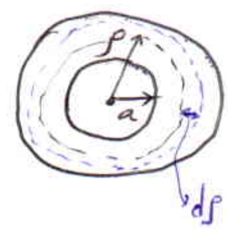
$d\Phi_i = \vec{B}_i \cdot d\vec{S}_e$ $\rightarrow$ flux enclosed in the region between ρ & $\rho+d\rho$ and the unit length in z-direction:

$d\Phi_i = B_i(\ell\, d\rho) = \frac{\mu_0 I \rho \ell}{2\pi a^2} d\rho \rightarrow \frac{d\Phi_i}{\ell} = \frac{\mu_0 I \rho}{2\pi a^2} d\rho$

$\Rightarrow \lambda_i = N\Phi_i \Rightarrow d\lambda_i = N\, d\Phi_i = \left(\frac{\pi\rho^2}{\pi a^2}\right)\frac{\mu_0 I \rho\, d\rho}{2\pi a^2} = \frac{\mu_0 I}{2\pi a^4}\rho^3 d\rho$

$\Rightarrow \lambda_i = \frac{\mu_0 I}{2\pi a^4}\int_0^a \rho^3 d\rho = \frac{\mu_0 I}{8\pi} \rightarrow \boxed{L_{\text{internal}} = \frac{\lambda_i}{I} = \frac{\mu_0}{8\pi} \quad H/m}$

$a \le \rho \le b$: $\oint_c \vec{B}_e \cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B}_e = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$



$d\Phi_e = B_e \cdot d\vec{S} = B_e \ell\, d\rho = \frac{\mu_0 I \ell}{2\pi\rho} d\rho \Rightarrow \frac{d\Phi_e}{\ell} = \frac{\mu_0 I\, d\rho}{2\pi\rho}$

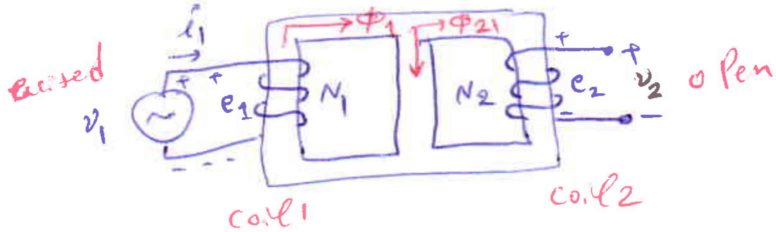
$d\lambda_e = N\, d\Phi_e = \left(\frac{I}{I}\right)\frac{\mu_0 I\, d\rho}{2\pi\rho} \Rightarrow \lambda_e = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{\rho} d\rho = \frac{\mu_0 I}{2\pi}\ln(b/a).$

$\Rightarrow \boxed{L_e = \frac{\lambda_e}{I} = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad H/m}$

$\rightarrow \boxed{L = L_i + L_e = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \ln(b/a)\right] \quad H/m}$

Mutual Inductance:

A m-circuit with 2 coils:



current i_1 in coil 1 etablishes Φ_1,

$$L_{11} = N_1 \frac{d\Phi_1}{di_1}$$

we are calculating coil 1 inductance.

The coil that is carrying the current & producing flux.

Φ_{21} = the fraction of Φ_1 that links coil 2.

induced emf in coil 2 $$e_2 = N_2 \frac{d\Phi_{21}}{dt} \rightarrow v_2 = N_2 \frac{d\Phi_{21}}{dt} \equiv L_{21} \frac{di_1}{dt}$$

mutual inductance $$\boxed{L_{21} = N_2 \frac{d\Phi_{21}}{di_1}}$$

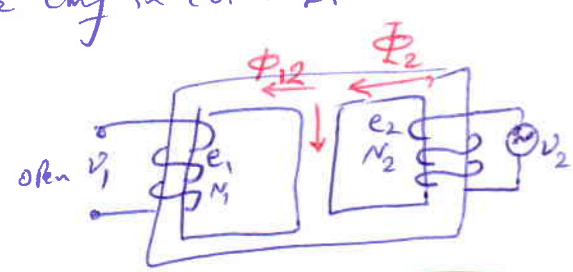
mutual inductance of coil 2 due to flux generated by coil 1.

mutual inductance between two coils = the total flux linking one coil per unit change in current in the other coil.

If we excite coil 2 with i_2 → produces Φ_2 while coil 1 is open, the self-inductance of coil 2: $$L_{22} = N_2 \frac{d\Phi_2}{di_2}$$

Φ_{12} = fraction of Φ_2 that links coil 1, inducede emf in coil 1:

$$e_1 = N_1 \frac{d\Phi_{12}}{dt}$$

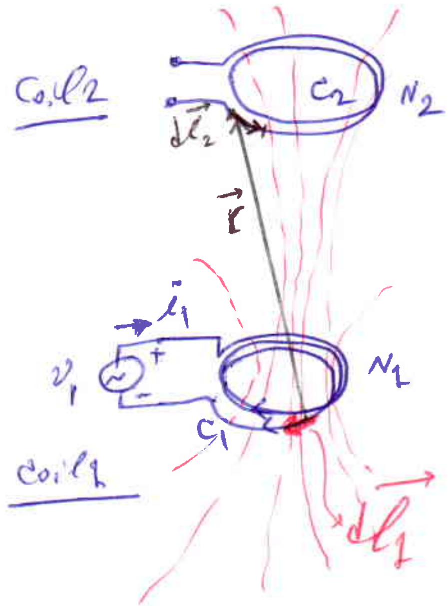


$$v_1 = N_1 \frac{d\Phi_{12}}{dt} = N_1 \frac{d\Phi_{12}}{di_2} \frac{di_2}{dt}$$

$$= \left(N_1 \frac{di_2}{dt}\right) \frac{d\Phi_{12}}{di_2} \equiv L_{12} \frac{di_2}{dt} \rightarrow \boxed{L_{12} = N_1 \frac{d\Phi_{12}}{di_2}}$$

mutual inductance of coil 1 due to flux by coil 2.

prove that $L_{12} = L_{21}$:



coil 1 is carrying a current,
the total flux linkages with coil 2 are

$$\lambda_{21} = N_2 \int_{S_2} \vec{B_1} \cdot d\vec{S_2}$$

$\vec{B_1}$ → flux density in the plane of coil 2 due to $i_1(t)$.

$$\rightarrow \lambda_{21} = N_2 \int_{S_2} (\vec{\nabla} \times \vec{A_1}) \cdot d\vec{S_2} \underset{\text{theorem}}{\overset{\text{stokes'}}{=}} N_2 \oint_{C_2} \vec{A_1} \cdot d\vec{l_2}$$

$$\vec{A_1} = \frac{\mu_1 N_1 i_1}{4\pi} \oint_{C_1} \frac{d\vec{l_1}}{r}$$

$$\rightarrow \lambda_{21} = \frac{\mu_1 N_1 N_2 i_1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l_1} \cdot d\vec{l_2}}{r} \quad ; \quad L_{21} = \frac{d\Phi_{21}}{d i_1}$$

$$\rightarrow \boxed{L_{21} = \frac{\mu_1 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l_1} \cdot d\vec{l_2}}{r}}$$

Neumann's formula for the inductance between 2 current-carrying coils.

If coil 2 is carrying $i_2(t)$ and 1 is ope: $\boxed{L_{12} = \frac{\mu_2 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l_1} \cdot d\vec{l_2}}{r}}$

→ For a linear m-medium like free space, $\mu_1 = \mu_2 \rightarrow \boxed{L_{12} = L_{21}}$

we do not use these 2 integrals, self- and mutual inductance are easier.

$\Rightarrow L_{12} = L_{21} = M$ mutual inductance between two coils.

if $\begin{cases} \Phi_{21} = k_1 \Phi_1 & , 0 \le k_1 \le 1 \\ \Phi_{12} = k_2 \Phi_2 & ; 0 \le k_2 \le 1 \end{cases} \rightarrow \boxed{M = N_1 \frac{d\Phi_{12}}{di_2} = N_1 \frac{d(k_2 \Phi_2)}{di_2} = N_1 k_2 \frac{d\Phi_2}{di_2}}$