Magnetic Vector Potential ($\vec{A}$):

$$\vec{\nabla}\cdot\vec{B}=0 \Longrightarrow \vec{B}=\vec{\nabla}\times\vec{A} \rightarrow \text{exist.} \left[\frac{Wb}{m}\right].$$

calculate $\vec{A}$: $\vec{B}=\frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{l}'\times\vec{R}}{R^3}$; $\vec{R}=(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}$

$$\vec{\nabla}\left(\frac{1}{R}\right)=-\frac{\vec{R}}{R^3} \Rightarrow \vec{B}=\frac{\mu_0 I}{4\pi}\int_C \vec{\nabla}\left(\frac{1}{R}\right)\times d\vec{l}'$$

{ primed: source coordinates
{ unprimed: field ~ ~ .

$$\vec{\nabla}\left(\frac{1}{R}\right)\times d\vec{l}'=\vec{\nabla}\times\left[\frac{d\vec{l}'}{R}\right]-\frac{1}{R}\left[\cancel{\vec{\nabla}\times d\vec{l}'}\right] \rightarrow 0$$

the curl operation is with respect to the unprimed coordinates of point $P(x,y,z) \Rightarrow \vec{\nabla}\times d\vec{l}'=0$

$$\vec{B}=\frac{\mu_0 I}{4\pi}\int_C \vec{\nabla}\times\left[\frac{d\vec{l}'}{R}\right]$$

$\rightarrow$ { $\int$ with respect primed; $\vec{\nabla}\times$ ~ ~ unprimed } $\rightarrow$ we can interchange the order.

$$\vec{B}=\vec{\nabla}\times\left[\frac{\mu_0}{4\pi}\int_C \frac{I\,d\vec{l}'}{R}\right] \Longrightarrow \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\boxed{\vec{A}=\frac{\mu_0}{4\pi}\int_C \frac{I\,d\vec{l}'}{R}}$$

If the wire is a closed loop: $\vec{A}=\frac{\mu_0}{4\pi}\oint_C \frac{I\,d\vec{l}'}{R} \Longrightarrow \boxed{\vec{A}=\frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v\,dv'}{R}}$

$\Rightarrow \vec{B}$ is defined since its curl & divergence is defined.

$\boxed{\vec{\nabla}\cdot\vec{A}=0 \rightarrow \text{Coulomb's gauge.}}$

m-flux $\Phi=\int_S \vec{B}\cdot d\vec{s}=\int_S(\vec{\nabla}\times\vec{A})\cdot d\vec{s} \xrightarrow[\text{theorem}]{\text{Stokes'}} \boxed{\Phi=\oint_C \vec{A}\cdot d\vec{l}}$

C is the contour bounding the open surface S.

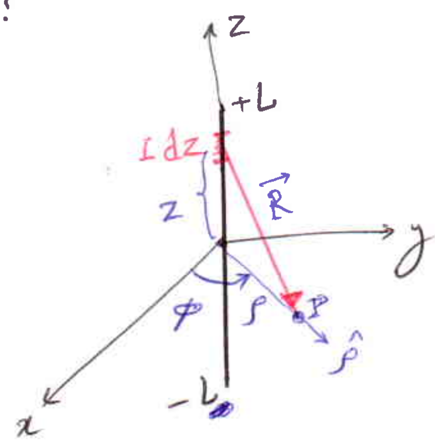
Example

EX: A straight conductor of length $2L$ located along the z-axis carries a current $\underline{I}$ in the z-direction. Find m-vector Potential at a Point in the bisecting Plane of the conductor. What is the m-flux density at that Point?

And for a very long ~~conductor~~ wire?

$\vec{R} = \vec{r} - \vec{r}\,'$

$\vec{R} = \rho\,\hat{\rho} - z\,\hat{z}$



$$\vec{A} = \frac{\mu_0}{4\pi}\,\hat{z}\int_{-L}^{L}\frac{I\,dz}{[\rho^2+z^2]^{1/2}}$$

$$= \frac{\mu_0 I}{4\pi}\left[\ln\left(L+\sqrt{L^2+\rho^2}\right) - \ln\left(-L+\sqrt{L^2+\rho^2}\right)\right]\hat{z}$$

For a very long wire $L \gg \rho \Rightarrow \begin{cases} L+\sqrt{L^2+\rho^2} \simeq L + L\left[1+\left(\frac{\rho}{2L}\right)^2\right] \simeq 2L \\ -L+\sqrt{L^2+\rho^2} \simeq -L + L\left[1+\left(\frac{\rho}{2L}\right)^2\right] \simeq \frac{\rho^2}{2L} \end{cases}$

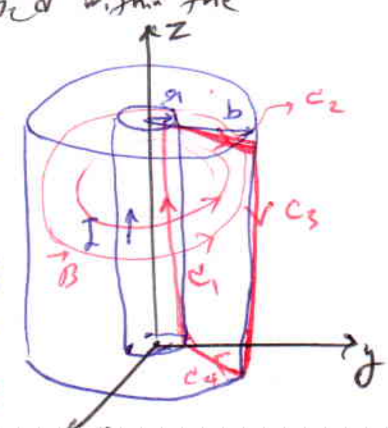
$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0 I}{2\pi}\ln\left[\frac{2L}{\rho}\right]\hat{z}}$$

$\vec{B}$ in P : $\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial \rho}\,\hat{\phi} =$

$$= \frac{\mu_0 I L}{2\pi\rho}\left[\frac{1}{\sqrt{L^2+\rho^2}}\right]\hat{\phi} \quad ; \text{for } L \gg \rho : \boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\,\hat{\phi}}$$

Example

EX: The inner conductor of a $\underline{100}$ m coaxial cable has a radius of 1 cm & carries 80 A current in z-direction. The outer conductor is very thin & has a radius of 10 cm. Calculate the total flux enclosed within the conductors.



$(b-a) \ll L \Rightarrow$ use the approximation $\vec{A} = \frac{\mu_0 I}{2\pi}\ln\left[\frac{2L}{\rho}\right]\hat{z}$

$$\Phi = \oint_{C} \vec{A}\cdot d\vec{l} = \sum_{i=1}^{4}\int_{C_i}\vec{A}_i\cdot d\vec{l}_i = \int_{C_1}\vec{A}_1\cdot d\vec{l}_1 + \cancel{\int_{C_2}\vec{A}_2\cdot d\vec{l}_2} + \int_{C_3}\vec{A}_3\cdot d\vec{l}_3 + \int_{C_4}\vec{A}_4\cdot d\vec{l}_4$$

0 only in $\hat{z}$

0 only in $\hat{z}$

$$\Rightarrow \Phi = \int_{c_1} \vec{A}_1 \cdot d\vec{l}_1 + \int_{c_3} \vec{A}_3 \cdot d\vec{l}_3 = \frac{\mu_0 I}{2\pi}\int_{-L}^{L} \ln\left(\frac{2L}{a}\right) dz - \frac{\mu_0 I}{2\pi}\int_{-L}^{L} \ln\left(\frac{2L}{b}\right) dz$$

$$= \frac{\mu_0 I L}{\pi}\ln\left(b/a\right).$$

$\rightarrow$ $I = 80\ A$, $a = 1\ cm$, $L = 50\ cm$, $b = 10\ cm$ $\rightarrow$ $\boxed{\Phi = 3.68\ mWb}$

Magnetic field Intensity & Ampère's Circuital law:

Like $\vec{D} = \epsilon \vec{E}$ which $\vec{D}$ was indep. of the permittivity of the medium.

$\Rightarrow$ define magnetic field intensity $\vec{H}$ in free space: $\vec{H} = \frac{\vec{B}}{\mu_0} \rightarrow \vec{B} = \mu_0 \vec{H}$

Ampère's law: $\oint_C \vec{H} \cdot d\vec{l} = I$ $\rightarrow$ enclosed current. = The net current intercepted by the area enclosed by the path.

Integral form.

- I could be a current carried by a conductor (any shape) or by flow of charges ($\bar{e}$ in a vacuum tube).

- If the current distribution has a high degree of symmetry, we can use Ampere's law to calculate m-field without laborious integration by Biot-Savart method.

$$I = \int_S \vec{J}_v \cdot d\vec{S} \Rightarrow \oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J}_v \cdot d\vec{S} \xrightarrow[\text{theorem}]{\text{stoke's}} \int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J}_v \cdot d\vec{S}$$

for any surface

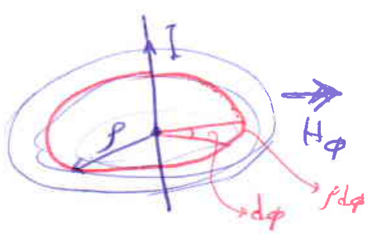
$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_v}$ the point (differential) form of Ampere's law.

Example

EX: wire, I, Z-direction. Find m-field intensity ($\vec{H}$).

$$\oint \vec{H} \cdot d\vec{l} = \int_0^{2\pi} H_\phi \rho\, d\phi = 2\pi\rho H_\phi$$

$$= I_{enc.} \Rightarrow \vec{H} = \frac{I}{2\pi\rho}\hat{\phi}.$$

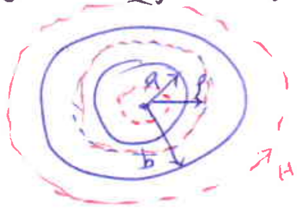


Example

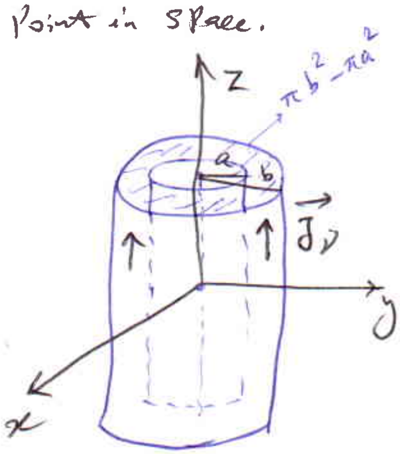
Ex: Hollow conductor, cylindrical (inner radius a, outer b) in z-axis, I,

If I is distributed uniformly, find m-field intensity at any point in space.

Ampere's circle:



$\vec{H} = H_\phi \hat{\phi}$



$$\vec{J}_v = \frac{I}{\pi(b^2-a^2)}\hat{z}$$

a) $\rho < a$: $\boxed{\vec{H} = 0}$, $I_{enc} = 0$

b) $a \le \rho \le b$: $\oint \vec{H}\cdot d\vec{l} = \int_s \vec{J}_v \cdot d\vec{s}$

$$= \frac{I}{\pi(b^2-a^2)} \int_a^\rho \int_0^{2\pi} \rho \, d\rho \, d\phi = \frac{I(\rho^2-a^2)}{b^2-a^2}$$

$$2\pi\rho H_\phi = \frac{I(\rho^2-a^2)}{b^2-a^2} \Rightarrow \boxed{\vec{H} = \frac{I}{2\pi\rho}\left[\frac{\rho^2-a^2}{b^2-a^2}\right]\hat{\phi}}$$

c) $\rho \ge a$: $2\pi\rho H_\phi = I \Rightarrow \boxed{\vec{H} = \frac{I}{2\pi\rho}\hat{\phi}}$

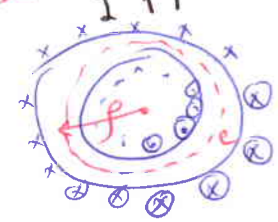
Example

Ex: Toroidal winding, N turns (ring), inner radius a, outer b, height h.

I, a) find m-field intensity within the ring ($\vec{H}$)

b) m-~~field~~ flux density ($\vec{B}$)

c) Total flux enclosed by the ring. Φ.



$\vec{H}$ in $\hat{\phi}$ direction and on radius ρ, is const.

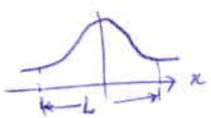
$$\oint \vec{H}\cdot d\vec{l} = I_{enc} \rightarrow H_\phi(2\pi\rho) = NI \Rightarrow \vec{H} = \frac{NI}{2\pi\rho}\hat{\phi}, \quad a \le \rho \le b$$

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 NI}{2\pi\rho}\hat{\phi}, \quad a \le \rho \le b$$

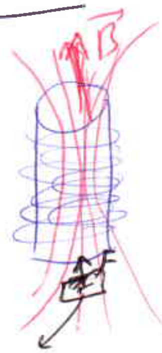
$$\Phi = \int \vec{B}\cdot d\vec{s} = \int \frac{\mu_0 NI}{2\pi\rho} d\rho \, dz = \frac{\mu_0 NI}{2\pi}\int_{\rho=a}^{b}\frac{d\rho}{\rho}\int_{z=0}^{h} dz = \frac{\mu_0 NIh}{2\pi}\ln(b/a).$$

Magnetic materials: extend theory of m-fields to regions containing m-materials

solenoid:



Force on these m-materials $\propto$ mass.
attraction or repulsion.



Repelled material: diamagnetic. (all organic compounds, & the majority of inorganic compounds).

→ diamagnetism: is a property of every atom & molecule.

Feeble Attraction: Paramagnetic (Al, cu,)

Ferromagnetic: The materials that are sucked strongly. $\simeq 5000$ (Paramag.)

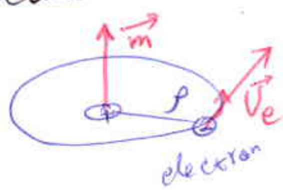
⇒ [Para + Dia] ≃ non magnetic materials.

* The permeability of all nonmagnetic materials is the same as free space.

→ Quantum mechanics: is needed to describe the magnetic properties of materials.

Simple calculation:

Atom



$$I = \frac{e\vec{V}_e}{2\pi\rho}$$

$$\vec{m}_0 = I\vec{A} = \frac{eV_e}{2\pi\rho}\cdot\pi\rho^2\hat{z} = \frac{eV_e\rho}{2}\hat{z}$$

① orbital m-moment.

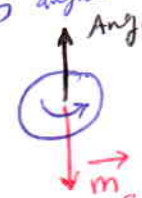
by Quantum mechanics: m is always some integral multiple of $\frac{h}{2\pi}$

h = Plank's const. ($h = 6.63\times10^{-34}$ J.S).

② Angular momentum: electron is continually rotating (Spinning) around its own axis, so provides angular momentum.

Angular momentum → rotational

$$\vec{L} = \vec{r}\times\vec{P} = \vec{r}\times m\vec{v}$$



spin magnetic moment.

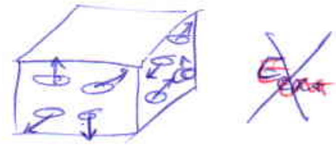
The spinning motion involves circulating charge → spin m-moment.

$$m_s = \frac{he}{8\pi m_e} = 9.27\times10^{-24}\ \text{A}\cdot\text{m}^2$$

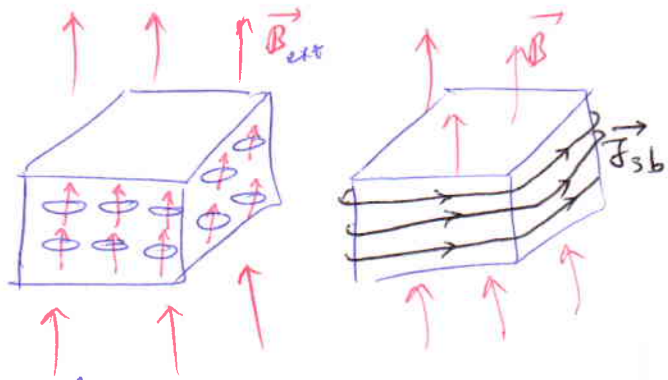
$\vec{m}_{net} = \vec{m}_{tot} = \vec{m}_{orb.} + \vec{m}_{spin}$ $\xRightarrow{\text{net } \vec{m}}$ Produces a far field like a current loop (magnetic dipole).

— In the absence of an external m-field : m-dipoles oriented randomly.

$\vec{m}_{net} \simeq 0$



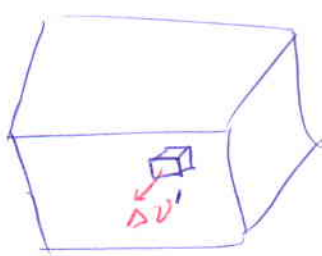
— In the presence of an external $\vec{B}_{ext}$: m-dipoles experiences a torques that try to align it with $\vec{B}_{ext}$.



* Main difference between electric dipole in a dielectric and m-dipoles in a magnetic material;

→ alignment of e-dipoles in a dielectric material always decreases the original e-field.
but the alignment of m-dipoles in Para- and Ferromagnetic materials Increases the original m-field. → this will create another $\vec{B}$ inside the m-material.

— n atom in Δv, $\vec{m}_i$ = m-moment of the i^{th} atom.



$\vec{M}$ = m-moment per unit volume.

$$= \lim_{\Delta v' \to 0} \frac{\sum_{i=1}^{n} \vec{m}_i}{\Delta v'}$$; if $\vec{M} \neq 0$: material is magnetized.

$d\vec{m} = \vec{M}\, dv' \;\rightarrow\; d\vec{A} = \dfrac{\mu_0 \vec{M} \times \hat{R}}{4\pi R^2}\, dv'$ → m-vector potential set up by $d\vec{m}$.

$\vec{\nabla}'\left(\frac{1}{R}\right) = \dfrac{\hat{R}}{R^2} \;\rightarrow\; d\vec{A} = \dfrac{\mu_0 \vec{M}}{4\pi} \times \vec{\nabla}'\left(\frac{1}{R}\right) dv'$

v' = volume of magnetized material.

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \vec{M} \times \vec{\nabla}'\left(\frac{1}{R}\right) dv'$$

; vector identity

$$\vec{M} \times \vec{\nabla}'\left(\frac{1}{R}\right) = \frac{1}{R}\vec{\nabla}' \times \vec{M} - \vec{\nabla}' \times \left[\frac{\vec{M}}{R}\right]$$

$$= \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{\nabla}' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi}\int_{v'} \vec{\nabla}' \times \left[\frac{\vec{M}}{R}\right] dv'$$

$$\int_{v'} \vec{\nabla}' \times \vec{M}\, dv' = -\oint_{s'} \vec{M} \times d\vec{s}' \quad ; \quad d\vec{s}' = ds'\,\hat{n}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{\nabla}' \times \vec{M}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{M} \times \hat{n}}{R} ds'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{J}_{vb}}{R} dv' + \frac{\mu_0}{4\pi}\int_{s'} \frac{\vec{J}_{sb}}{R} ds'.$$

$\rightarrow \begin{cases} \vec{J}_{vb} = \vec{\nabla} \times \vec{M} & \rightarrow \text{bound vol current density} \\ \vec{J}_{sb} = \vec{M} \times \hat{n} & \rightarrow \text{bound surface current density.} \end{cases}$

* drop prime: $\vec{\nabla}\times$ and $\times$ operations refer to the coordinates of the source point.

→ We can use $\vec{J}_{vb}$ & $\vec{J}_{sb}$ to calculate the m-vector pot. due to magnetized material.

→ If we have free vol. current density $\vec{J}_{vf}$ & free surface c.d. $\vec{J}_{sf}$:

$\Rightarrow \vec{J}_v = \vec{J}_{vf} + \vec{J}_{vb}$

but $\vec{J}_{vf} = \vec{\nabla} \times \vec{H}$, $\vec{B} = \mu_0 \vec{H}$ or $\vec{H} = \dfrac{\vec{B}}{\mu_0}$ $\rightarrow \boxed{\vec{\nabla} \times \left[\frac{\vec{B}}{\mu_0}\right] = \vec{J}_{vf}}$ free space.

medium: $\vec{\nabla} \times \left[\dfrac{\vec{B}}{\mu_0}\right] = \vec{J}_{vf} + \vec{J}_{vb} = \vec{\nabla} \times \vec{H} + \vec{\nabla} \times \vec{M}$ $\rightarrow \boxed{\vec{B} = \mu_0\left[\vec{H} + \vec{M}\right]}$

general

for Linear, homogeneous & isotropic medium: $\vec{M} = \chi_m \vec{H}$

χ_m → m-susceptibility.

$$\vec{B} = \mu_0 [1 + \chi_m] \vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H}$$

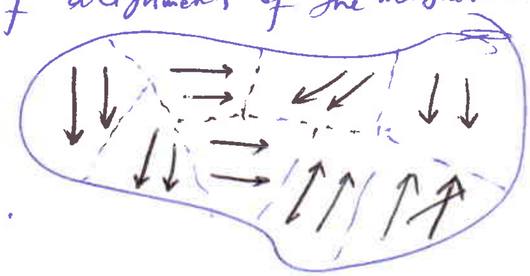
μ_r → relative permeability.

μ → Permeability of medium

$$\begin{cases} \mu_r = 1 & \text{for Para- \& diamagnetic materials.} \\ \mu_r \longrightarrow 5000 & \text{ferromagn. for } 1\,T. \end{cases}$$

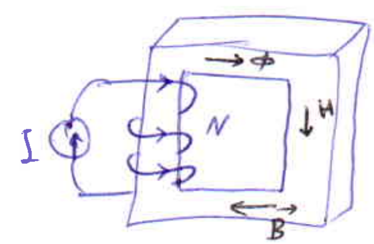
Ferromagnetism:

Magnetic domains: a small region in which all the magnetic dipoles are perfectly aligned. The direction of alignment of the magnetic dipoles varies from one domain to the next.



→ Virgin material in nonmagnetized state.

magnetization characteristic of the magnetic materials:



By winding a wire around a m-material and let I go through the wire, we can create $\vec{H}$, then it will create $\vec{B}$ field within the medium. → As long as $\vec{B}$ is weak, the movement of the domain walls is reversible.

$\vec{H} \uparrow \Rightarrow \vec{B} \uparrow \Rightarrow$ more m-dipoles align themselves with the $\vec{B}$-field.

→ changes in $\vec{B}$ is due to changes in $\vec{M}$.

$$\vec{M} = \frac{\vec{B}}{\mu_0} - \vec{H}$$ → knowing $\vec{B}$ & $\vec{H}$ we can determine $\vec{M}$.

- As we decrease the current I, $\vec{H}$ will decrease but $\vec{B}$ does not decrease as fast. hysteresis: This irreversibility is called hysteresis.

$\vec{H}=0 \rightarrow \vec{B} = \vec{B_r}$: residual, or remanent flux density.

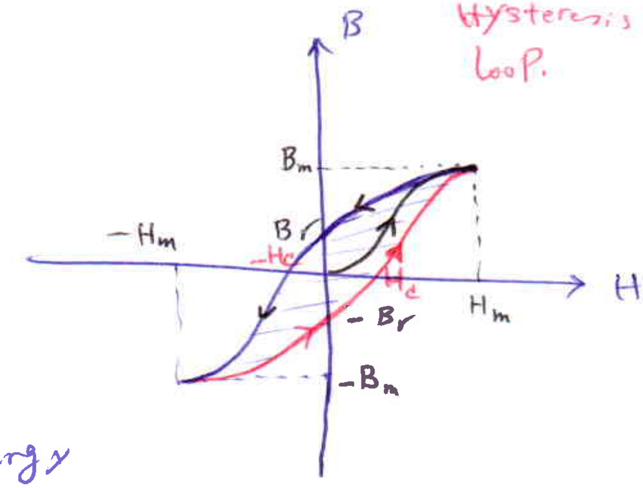
↘ Permanent magnet.

m-material in direct-current machines: use material that leaves more residual flux density. (Hard m-materials).

- reverse the direction of I:

$\vec{H}$ in opposite direction,

H_c = Coersive force: $|\vec{H_c}|$ at which $\vec{B}=0$



→ The area of the hysteresis loop is the energy lost per cycle. (hysteresis loss). → we need this energy to align the m-domains in one direction & then realign in opposite direction once per cycle.

→ For Alternating current application, we need Less hysteresis loss → less residual flux density.

→ we need soft magnetic materials.

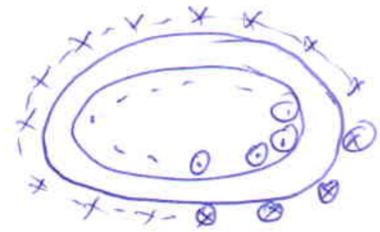
Example

EX: Toroidal winding, N turns, ring (inner $\underline{a}$, outer $\underline{b}$), around magnetic material with relative permeability μ_r, Find:

a) m-moment per unit volume

b) the bound volume current density

c) " " surface " ".

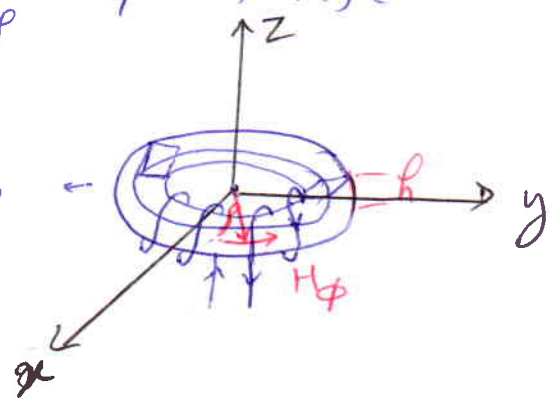


m-susceptibility $\chi_m = \mu_r - 1$

$\vec{H} = \frac{NI}{2\pi\rho}\hat{\phi}$; $a \le \rho \le b$

m-moment per unit volume $\equiv$ magnetization vector $\vec{M} = \chi_m \vec{H} = \frac{(\mu_r - 1)NI}{2\pi\rho}\hat{\phi}$; $a \le \rho \le b$.

$\rightarrow \vec{J}_{vb} = \vec{\nabla} \times \vec{M} = 0$



4 surfaces $\leftarrow$

Top surface $\vec{J}_{sb|tops.} = \vec{M} \times \hat{z}$

$$= \frac{(\mu_r - 1)NI}{2\pi\rho}\hat{\rho}$$

bottom s. $\vec{J}_{sb|bottoms.} = \vec{M} \times (-\hat{z}) = -\frac{(\mu_r - 1)NI}{2\pi\rho}\hat{\rho}$

surface at $\rho = a$: $\vec{J}_{sb|\rho=a} = \vec{M} \times (-\hat{\rho}) = \frac{(\mu_r - 1)NI}{2\pi a}\hat{z}$

" , $\rho = b$: $\vec{J}_{sb|\rho=b} = \vec{M} \times \hat{\rho} = -\frac{(\mu_r - 1)NI}{2\pi b}\hat{z}$ ✓ like what it should be.

Magnetic scalar Potential:

$\vec{A}$ = m-Vector Potential $\rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$; $\vec{A} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}_v\, dv}{R}$; $\vec{\nabla} \times \vec{H} = \vec{J}_v$

in a current-carrying region the m-field intensity is not conservative. $\Leftarrow$ $\boxed{\vec{H}\text{ field is rotational.}}$ $\xleftarrow{\text{but}}$ $\vec{\nabla} \times \vec{E} = 0$

e-field intensity at any point due to fixed charges always co...