Magnetostatics

Discovery: Permanently magnetized iron ore, lodestone.

Magnetic Force: lodestone orient itself in the North & South direction →
led to existance of another force (magnetic force).

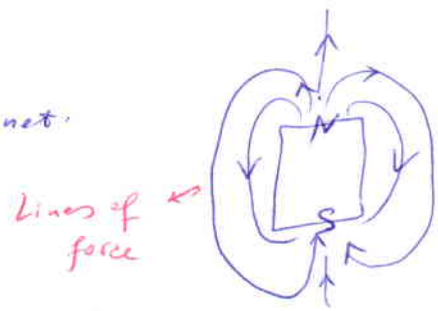
magnetic material: Any material that can be influenced (magnetized) by the magnetic force. (iron, cobalt, nickel).

magnetized (magnet).
(magnetite)

{ North (Seeking) Pole
{ South (Seeking) Pole → A freely suspended magnet.

Magnetic Field: A m-field is associated with a magnet.

N N } repel , NS } attract.
S S }   SN }



→ N & S cannot be separated. → isolated magnetic pole is not a physical reality.

Orsted exp.: A magnetic needle was deflected by a current in a wire.

electricity ⟹ magnetism.

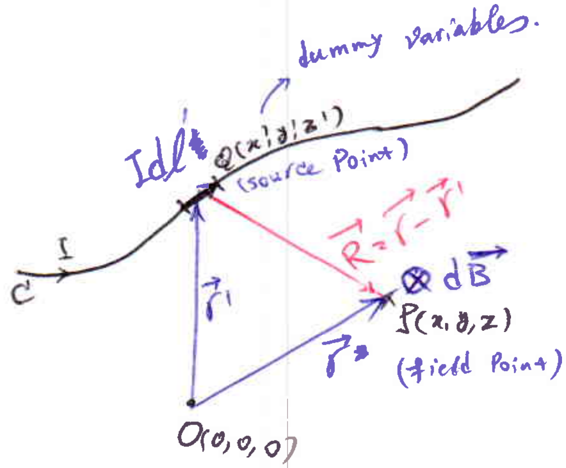
⇒ electric currents are also sources of m-fields.

Biot & Savart: experimentally formulated an eqn for "magnetic flux density" at a point produced by a current-carrying conductor. ≡ coulomb's law.

Ampere: m-force between current wires. ⇒ electric machines.

Magnetostatics: m-fields produced by steady currents.

The Biot-Savart law:



$$d\vec{B} = \underset{\text{const.}}{k}\, \frac{I d\vec{l} \times \hat{R}}{R^2}$$

$d\vec{B}$: elemental m-flux density (Tesla)

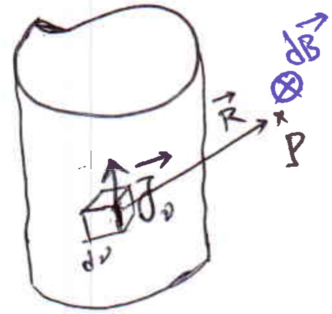
$1\,T = 1\,\frac{Wb}{m^2}$

$\vec{R} = \vec{r} - \vec{r}'$ $(Q \to P)$

$k = \frac{\mu_0}{4\pi}$; $\mu_0 = (4\pi)10^{-7}$ H/m

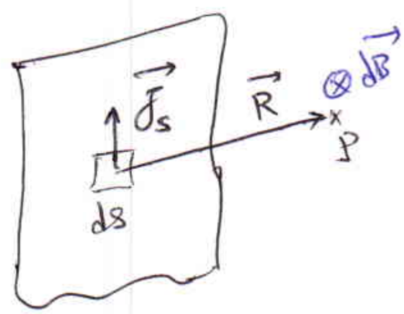
μ_0: free space Permeability

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{R}}{R^3} \Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int_C \frac{I d\vec{l} \times \vec{R}}{R^3}.$$



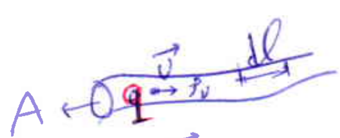
$$I\, d\vec{l} = \vec{J}_v\, dv \Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int_v \frac{\vec{J}_v \times \vec{R}}{R^3}\, dv.$$

$$I\, d\vec{l} = \vec{J}_s\, ds \rightarrow \vec{B} = \frac{\mu_0}{4\pi} \int_s \frac{\vec{J}_s \times \vec{R}}{R^3}\, ds.$$



If charge q is moving with an average velocity of $\vec{U}$, ρ_v = vol. charge density

A = cross-section of wire



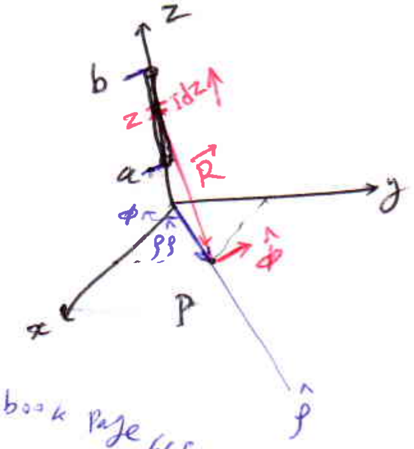
$$dq = \rho_v \underbrace{(A\, dl)}_{dv} \Rightarrow I = \frac{dq}{dt} = \rho_v A \frac{dl}{dt} = \rho_v A \vec{U} \rightarrow \frac{I}{A} = \vec{J} \rightarrow \vec{J}_v = \rho_v \vec{U}$$

$$\rightarrow \vec{J}_v\, dv = \rho_v \vec{U}\, dv = dq\, \vec{U}$$

$$\rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi} \left[\frac{q\vec{U} \times \vec{R}}{R^3}\right]}$$

$\vec{B}$ of a q moving with $\vec{U}$ at a distance $\vec{R}$.

EX: A wire (filamentary), finite length from $z=a$ to $z=b$. find $\vec{B}$ at P in xy plane? what if $a\to-\infty$ & $b\to\infty$?



$$d\vec{B}=\frac{\mu_0 I\, d\vec{l}\times\vec{R}}{4\pi\, R^3} \quad ; \quad I d\vec{l}=I\,dz\,\hat{z} \rightarrow I d\vec{l}\times\vec{R}=I\rho\,dz\,\hat{\phi}$$

$$\vec{r}-\vec{r}'=\vec{R}=\rho\hat{\rho}-z\hat{z}$$

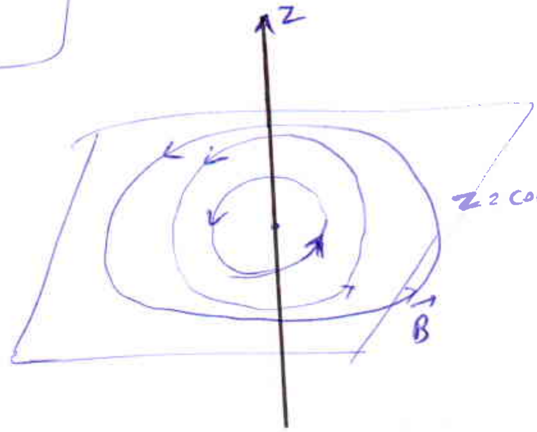
$$\vec{B}=\frac{\mu_0 I\rho}{4\pi}\int_a^b \frac{dz}{[\rho^2+z^2]^{3/2}}\hat{\phi}=\frac{\mu_0 I\rho}{4\pi}\frac{1}{\rho^2}\frac{z}{\sqrt{\rho^2+z^2}}\Bigg|_a^b$$

book page 665 #6 of $\int$.

$$\boxed{\vec{B}=\frac{\mu_0 I}{4\pi\rho}\left[\frac{b}{\sqrt{\rho^2+b^2}}-\frac{a}{\sqrt{\rho^2+a^2}}\right]\hat{\phi}}$$

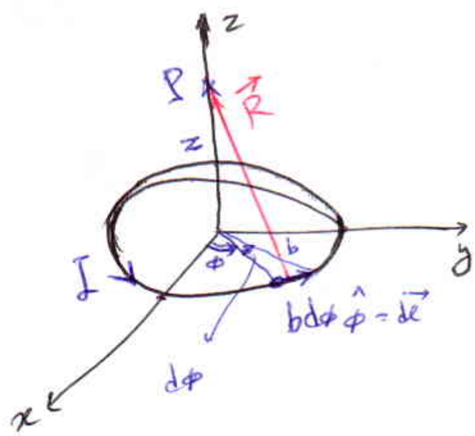
$$\text{if}\begin{cases}a\to-\infty\\ b\to+\infty\end{cases}\Rightarrow \boxed{\vec{B}=\frac{\mu_0 I}{2\pi\rho}\hat{\phi}}$$

$$\vec{B}\propto\frac{1}{\rho}\equiv\vec{E}\propto\frac{1}{r}$$



EX: A circular loop (in xy plane), I. $\vec{B}$ on +z axis?

what if $z\to\infty$?



$$d\vec{B}=\frac{\mu_0 I\, d\vec{l}\times\vec{R}}{4\pi\, R^3} \quad ; \quad d\vec{l}=b\,d\phi\,\hat{\phi}$$

$$\vec{R}=\vec{r}-\vec{r}'=z\hat{z}-\rho\hat{\rho}=z\hat{z}-b\hat{\rho}$$

$$d\vec{l}\times\vec{R}=(b^2\hat{z}+bz\hat{\rho})\,d\phi$$

$$\vec{B}=\frac{\mu_0 I b^2}{4\pi}\int_0^{2\pi}\frac{d\phi}{(b^2+z^2)^{3/2}}\hat{z}+\frac{\mu_0 I b z}{4\pi}\int_0^{2\pi}\frac{\hat{\rho}\,d\phi}{(b^2+z^2)^{3/2}} \xrightarrow{} 0 \quad ; \hat{\rho}=\cos\phi\,\hat{x}+\sin\phi\,\hat{y}$$

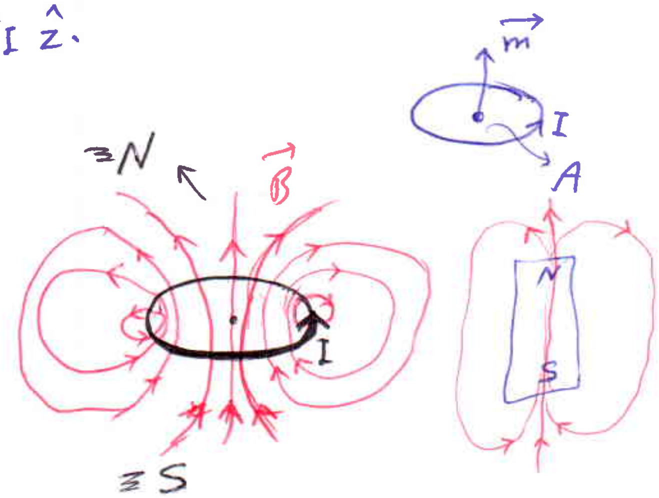
$$\boxed{\vec{B}=\frac{\mu_0 I b^2}{2(b^2+z^2)^{3/2}}\hat{z}} \quad \text{if } z=0 \rightarrow \vec{B}=\frac{\mu_0 I}{2b}\hat{z}$$

$$\text{if } z\to\infty \rightarrow (b^2+z^2)^{3/2}\simeq z^3 \rightarrow \vec{B}=\frac{\mu_0 I b^2}{2z^3}\hat{z} \rightarrow \text{magnetic dipole.}$$

83

magnetic dipole moment: $\vec{m} = AI\,\hat{z} = \pi b^2 I\,\hat{z}$.

$\Rightarrow \boxed{\vec{B} = \dfrac{\mu_0 \vec{m}}{2\pi z^3}}$

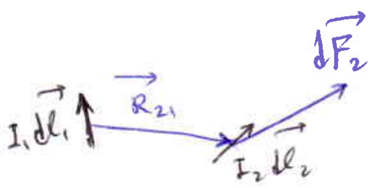


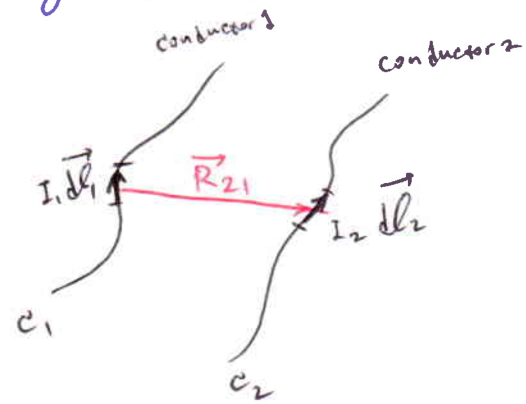
$\Rightarrow$ A current carrying coil forms an electromagnet!

Ampère's force law: Force between two current carrying conductors:

Force exerted on (2) by (1):

Ampère found experimentally:





$$d\vec{F}_2 = \frac{\mu_0 I_2 d\vec{l}_2}{4\pi} \times \left[\frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3}\right]$$

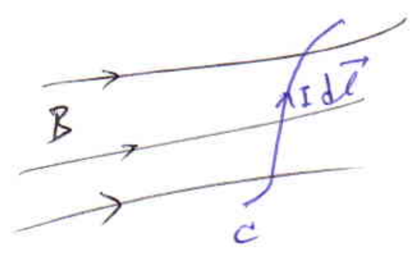
$$\boxed{\vec{F}_2 = \frac{\mu_0}{4\pi}\int_{c_2} I_2 d\vec{l}_2 \times \underbrace{\int_{c_1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3}}_{\vec{B}_1}}$$

$$\vec{F}_2 = \int_{c_2} I_2 d\vec{l}_2 \times \vec{B}_1$$

$\hookrightarrow$ m-field of 1 at $I_2 d\vec{l}_2$ location

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\int_{c_1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3}$$

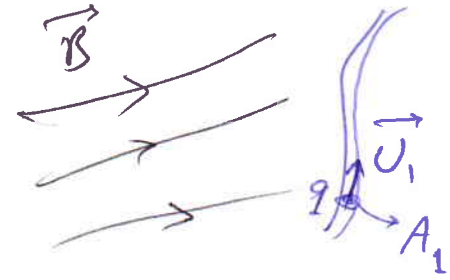
$\Rightarrow$ If a current-carrying conductor is placed in an external $\vec{B}$, the force experienced by the conductor is: $\boxed{\vec{F} = \int_c I\, d\vec{l} \times \vec{B}}$



general: $\Downarrow$

$$\boxed{\vec{F} = \int_v \vec{J}_v \times \vec{B}\, dv} \quad \text{or} \quad \boxed{\vec{F} = \int_s \vec{J}_s \times \vec{B}\, ds}$$

force on moving charge by external $\vec{B}$:

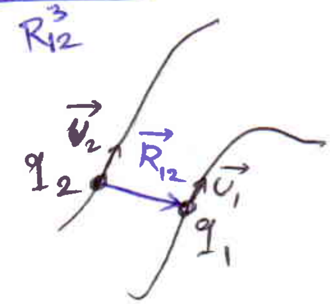


$\rho_{v_1}, \vec{U}_1, A_1 \Rightarrow dq_1 = \rho_{v_1} A_1 \, dl_1 \quad ; J_{v_1} = \dfrac{dq_1/dt}{A}$

$\vec{J}_{v_1} \, dv_1 = dq_1 \vec{U}_1 \quad \leftarrow \quad = \rho_{v_1} \dfrac{dl_1}{dt} = \rho_{v_1} \vec{U}_1$

$\Rightarrow \boxed{\vec{F}_1 = q_1 \vec{U}_1 \times \vec{B}}$ force on q_1 by $\vec{B}$.

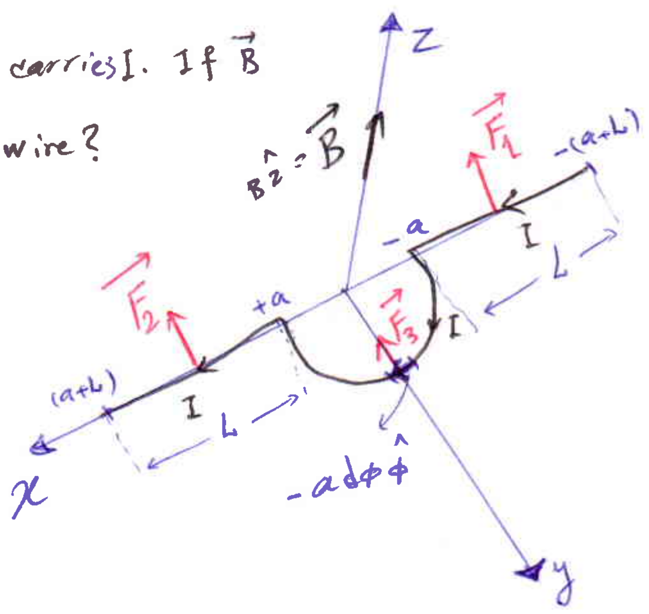
If $\vec{B}$ is created by another moving charge: $\vec{B} = \dfrac{\mu_0}{4\pi} \dfrac{q_2 \vec{U}_2 \times \vec{R}_{12}}{R_{12}^3}$



$$\boxed{\vec{F}_1 = \frac{\mu_0}{4\pi R_{12}^3} \left[q_1 \vec{U}_1 \times q_2 \vec{U}_2 \times \vec{R}_{12} \right]}$$

could be the fundamental law of magn. force. $\xrightarrow{\text{to}}$ obtain Ampere & Biot-Savart law.

EX: A half-circle wire lines in the xy plane & carries I. If $\vec{B}$ in the region is $\vec{B} = B\hat{z}$, what is m-force on wire?



$\vec{F} = \int_C I \, d\vec{l} \times \vec{B} \qquad ; \begin{cases} d\vec{l}_1 = dl_1 \hat{x} = dx \, \hat{x} \\ d\vec{l}_2 = dl_2 \hat{x} = dx \, \hat{x} \end{cases}$

$\vec{F}_1 = \int_{-(a+L)}^{-a} I \, dx \, \hat{x} \times B \hat{z} = -BIL \hat{y}$

$\vec{F}_2 = \int_{a}^{a+L} I \, dx \, \hat{x} \times B\hat{z} = -BIL\hat{y}$

$\leftarrow d\vec{l}_3 = -a \, d\phi \, \hat{\phi}$

$\vec{F}_3 = \int_{\pi}^{0} IB(-\hat{\phi} \times \hat{z}) \, a \, d\phi$

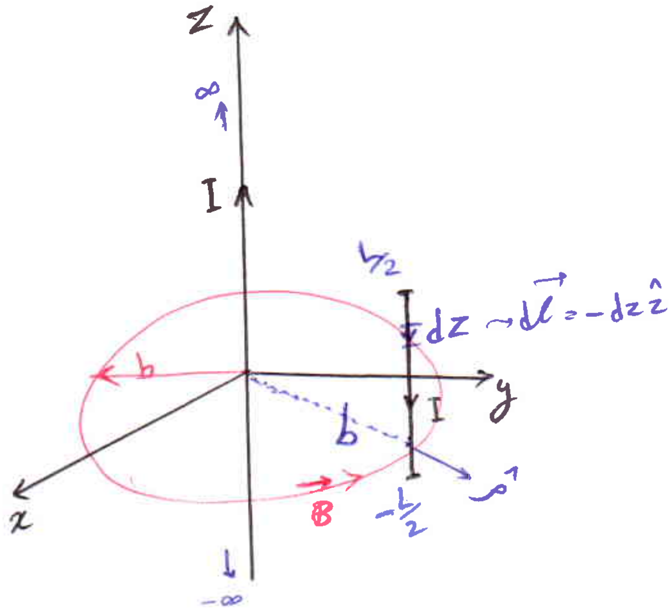
$= -\int_{\pi}^{0} \hat{\rho} \, BIa \, d\phi = BIa \int_0^{\pi} [\hat{x} \cos\phi + \hat{y} \sin\phi] \, d\phi = -2IBa\,\hat{y} \rightarrow$ Like straight wire. $L = 2a$!

$\Rightarrow \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = -2IB(a+L)\hat{y}$

EX: Force between two wires:

$\vec{B}$ of an infinite wire carrying I at distance b

is: $\vec{B} = \frac{\mu_0 I}{2\pi b} \hat{\phi}$



$$\vec{F} = \int_c I\, d\vec{l} \times \vec{B}$$

$$= \frac{\mu_0 I^2}{2\pi b} \int_{-L/2}^{L/2} (-\hat{z} \times \hat{\phi})\, dz$$

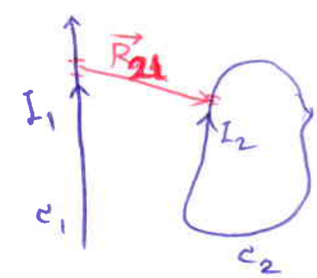
repulsive.

$$= -\frac{\mu_0 I^2}{2\pi b} \int_{-L/2}^{L/2} \hat{\rho}\, dz = \frac{\mu_0 I^2 L}{2\pi b} \hat{\rho} \Rightarrow \frac{\vec{F}}{L} = \frac{\mu_0}{2\pi b} I^2 \hat{\rho} \quad [N/m].$$

used to define the unit of current (A).

1 A ⇒ when two parallel wires of length 1^m, separated by 1^m, and $F = 2\times10^{-7} N$ ⇒ The current through each conductor is 1 A.

The m-force experienced by a current-carrying loop of arbitrary shape:



$$\vec{F}_2 = \frac{\mu_0}{4\pi} \int_{c_2} I_2 d\vec{l}_2 \times \int_{c_1} \frac{I_1 d\vec{l}_1 \times \vec{R}_{21}}{R_{21}^3}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \int_{c_2}\int_{c_1} \frac{1}{R_{21}^3} (\underset{A}{d\vec{l}_2} \times \underset{B}{d\vec{l}_1} \times \underset{C}{\vec{R}_{21}})$$

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \Rightarrow$

$$\vec{F}_2 = \frac{\mu_0 I_1 I_2}{4\pi} \left[\int_{c_2}\int_{c_1} \frac{d\vec{l}_2 \cdot \vec{R}_{21}}{R_{21}^3} d\vec{l}_1 - \int_{c_2}\int_{c_1} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R_{21}^3} \vec{R}_{21} \right]$$

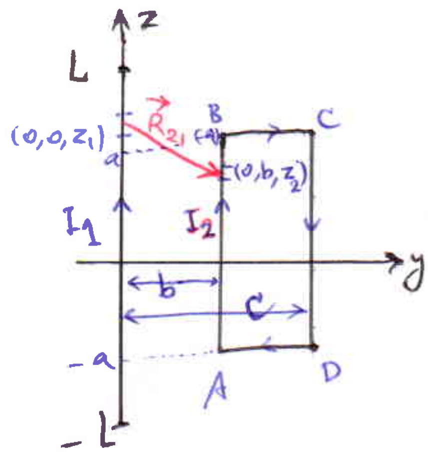
$$, \frac{\vec{R}_{21}}{R_{21}^3} = -\vec{\nabla}\left(\frac{1}{R_{21}}\right) \Rightarrow -\int_{c_2}\int_{c_1} \left[\vec{\nabla}\left(\frac{1}{R_{21}}\right) \cdot d\vec{l}_2\right] d\vec{l}_1 \underset{\text{for closed loop}}{\overset{\text{Stokes' theorem}}{=}} -\int_{c_1}\int_{S_2} \left[\vec{\nabla} \times \vec{\nabla}\left(\frac{1}{R_{21}}\right) \cdot d\vec{S}_2\right] d\vec{l}_1$$

$\therefore$ but curl of a gradient of a scalar function is always zero.

$$\Rightarrow \vec{F}_2 = -\frac{\mu_0 I_1 I_2}{4\pi} \int_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R_{21}^3} \vec{R}_{21} .$$

EX: A rectangular loop carrying I_2 is placed close to a straight wire I_1. $\vec{F}$ experienced by loop?

forces on AB, BC, CD, DA.



$$d\vec{l}_1 = dz_1\, \hat{z}$$

$$d\vec{l}_2 = \begin{cases} dz_2\, \hat{z} & \text{for } AB \,\&\, CD \\ dy_2\, \hat{y} & \text{for } BC \,\&\, AD \Rightarrow d\vec{l}_1 \cdot d\vec{l}_2 = 0 \end{cases}$$

$$\vec{R}_{21} = (b-0)\hat{y} + (z_2 - z_1)\hat{z}$$

$$\vec{F}_{AB} = -\frac{\mu_0 I_1 I_2}{4\pi} \int_{-L}^{L} dz_1 \int_{-a}^{a} \frac{b\hat{y} + (z_2 - z_1)\hat{z}}{\left[b^2 + (z_2 - z_1)^2\right]^{3/2}} dz_2$$

$$= \underbrace{-\frac{\mu_0 I_1 I_2}{2\pi b}}_{(-)} \left[\underbrace{\sqrt{(L+a)^2 + b^2} - \sqrt{(L-a)^2 + b^2}}_{(+)}\right] \hat{y}$$

$\Rightarrow$ − : attractive.

$$\vec{F}_{CD} = \underbrace{\frac{\mu_0 I_1 I_2}{2\pi c}}_{(+)} \left[\underbrace{\sqrt{(L+a)^2 + c^2} - \sqrt{(L-a)^2 + c^2}}_{(+)}\right] \hat{y}$$

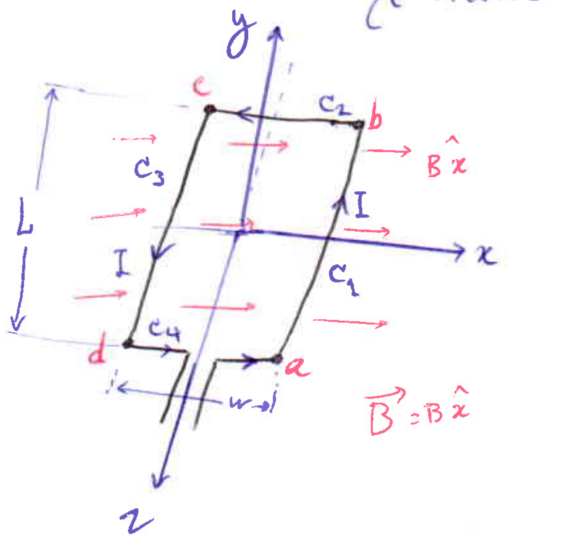
$\Rightarrow$ + : repulsion.

$$\vec{F}_{tot} = -\hat{y} \frac{\mu_0 I_1 I_2}{2\pi} \left[\frac{1}{b}\left[\sqrt{(L+a)^2 + c^2} - \sqrt{(L-a)^2 + b^2}\right] - \frac{1}{c}\left[\sqrt{(L+a)^2 + c^2} - \sqrt{(L-a)^2 + c^2}\right]\right].$$

since $c > b \Rightarrow \vec{F}_{tot}$ is attractive.

Magnetic torque: A conductor in $\vec{B}$ $\Rightarrow$ move in a direction $\perp$ to B & $\vec{l}$.

a current carrying loop in $\vec{B}$: m-force can impart a rotation to the coil. $\Rightarrow$ $\begin{cases} \text{e-motors.} \\ \text{e-meters.} \end{cases}$

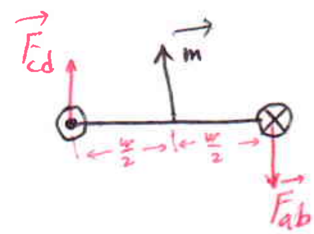


$A = LW$ cross-sectional area.

→ No force on bc & da.

$$\begin{cases} \vec{F}_{ab} = -BIL\hat{y} \\ \vec{F}_{cd} = BIL\hat{y} \end{cases}$$

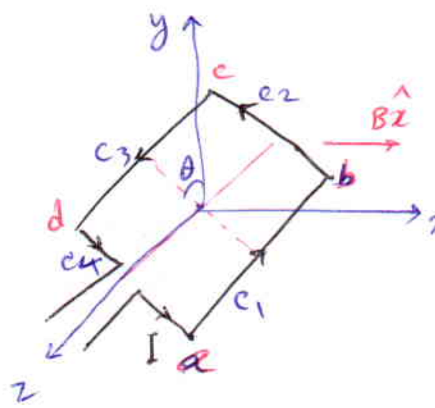
→ Lines of action of the two forces do not coincide, these forces exert a torque, which tends to rotate the coil about z-axis.



$$\begin{cases} \vec{\tau}_{ab} = \frac{\vec{w}}{2} \times \vec{F}_{ab} = \frac{w}{2}\hat{x} \times (-BIL\hat{y}) = -\frac{1}{2}BILw\,\hat{z} \\ \vec{\tau}_{cd} = \frac{\vec{W}}{2} \times \vec{F}_{cd} = \frac{w}{2}(-\hat{x}) \times (BIL\hat{y}) = -\frac{1}{2}BILw\,\hat{z} \end{cases}$$

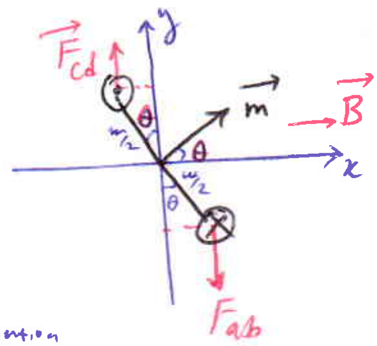
$$\vec{\tau}_{tot} = \vec{\tau}_{ab} + \vec{\tau}_{cd} = -BILw\,\hat{z} = -BIA\,\hat{z} = IA\,\hat{y} \times B\hat{x} = \vec{m} \times \vec{B}$$

$$\boxed{\begin{array}{l} \vec{m} = IA\,\hat{y} \\ \vec{\tau} = \vec{m} \times \vec{B} \end{array}}$$



⇒ coil rotates under the influence of torque.

$$\begin{cases} \vec{F}_{bc} = \int_{c_2} I(dx\,\hat{x} + dy\,\hat{y}) \times B\hat{x} = -IB\int_{-\frac{w}{2}\cos\theta}^{\frac{w}{2}\cos\theta} dy\,\hat{z} = -BIW\cos\theta\,\hat{z} \\ \\ \vec{F}_{da} = BIW\cos\theta\,\hat{z} \end{cases}$$



→ but the line of action of $\vec{F}_{bc}$ & $\vec{F}_{da}$ is the same ⇒ $\vec{F}_{bc} + \vec{F}_{da} = 0$ → no contribution

$$\begin{cases} \vec{T}_{ab} = \frac{w}{2}[\sin\theta\,\hat{x} + \cos\theta\,\hat{y}] \times (-BIL\hat{y}) = -\frac{1}{2}BILw\sin\theta\,\hat{z} \\ \vec{T}_{cd} = \frac{w}{2}[\sin\theta\,(-\hat{x}) + \cos\theta\,\hat{y}] \times (BIL\hat{y}) = -\frac{1}{2}BILw\sin\theta\,\hat{z} \end{cases}$$

$$\vec{T}_{tot} = \vec{T}_{ab} + \vec{T}_{cd} = -BI\underbrace{Lw}_{A}\sin\theta\,\hat{z} = \vec{m} \times \vec{B}$$

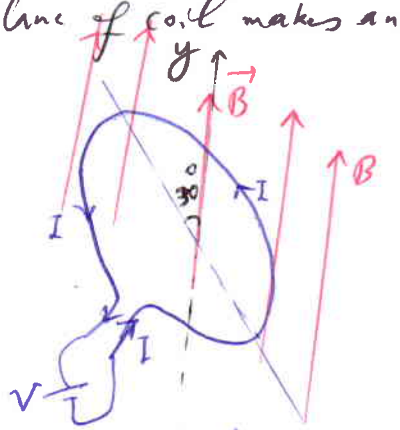
$$= I\vec{A} \times B\hat{x} = ILW\sin\theta\,\hat{y} \times B\hat{x}.$$

Torque experienced by coil varies sinusoidally (T_{max} when $\vec{A} \perp \vec{B}$, plane of coil is parallel to $\vec{B}$).

- when $\vec{m}$ & $\vec{B}$ align with each other, no torque is remained, coil is locked in that position.

→ if coil has $\underline{N}$ times of loop → $\vec{T}_{tot} = N\vec{T}_1$.

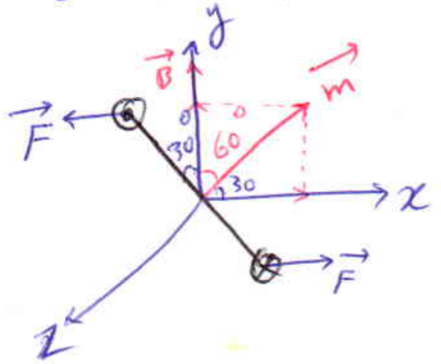
EX: A circular loop of 200 turns, mean area of $10\ cm^2$. Plane of coil makes an angle of 30° with $\vec{B} = 1.2\,T$. if $I = 50\,A$ → $T_{tot} = ?$



$$m = NIA = (200)(50)(10\times10^{-4}) = 10\ [A.m^2].$$

$$\vec{T} = \vec{m} \times \vec{B} = (m\cos30\,\hat{x} + m\sin30\,\hat{y}) \times (1.2\,\hat{y})$$

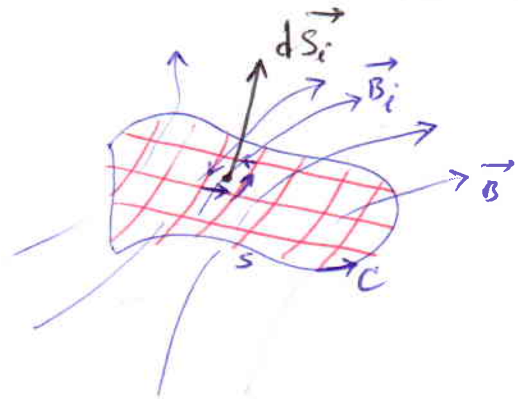
$$= \hat{z}\,(10\cos30)(1.2) = 10.39\,\hat{z}\ [N.m].$$



Gauss's law for m-field & magnetic flux:

$\Delta\Phi_i = \vec{B}_i \cdot \Delta\vec{S}_i$

magnetic flux passing through $\Delta\vec{S}_i$.



$\phi = \sum_{i=1}^{n} \vec{B}_i \cdot \Delta\vec{S}_i \rightarrow \boxed{\Phi = \int_s \vec{B} \cdot d\vec{S}}$

[Wb].

$\boxed{\oint_s \vec{B} \cdot d\vec{S} = 0}$ → North Pole & South Pole are not separable. → Lines of $\vec{B}$ are always continuous.
$\vec{B}$ of a conductor are circles.

integral form of Gauss's law for m-fields.

$\oint_s \vec{B} \cdot d\vec{S} = \int_v \vec{\nabla} \cdot \vec{B}\, dv = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$ diff. form of Gauss's law for m-fields.

⇐ maxwell's eqn. ⇐ $\vec{B}$ is solenoidal (continuous).

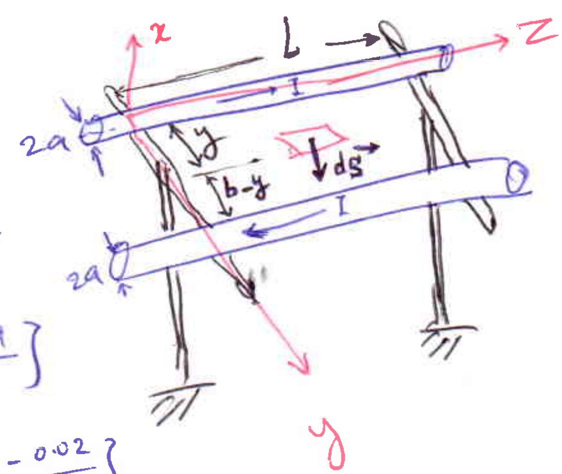
Two wire transmission line:

EX: Two very long parallel wire with 1000 A in opposite directions, 1 m apart.
radius of wire = a; they are strung on poles 100 m apart. determine
Flux passing through the region bounded by conductors & two consecutive Poles.

$a = 0.02\,m,\ b = 1\,m,\ L = 100\,m,\ I = 1000\,A.$

$\vec{B} = \frac{\mu_0 I}{2\pi y}(-\hat{x}) + \frac{\mu_0 I}{2\pi(b-y)}(-\hat{x})$

$= \frac{-\mu_0 I}{2\pi}\left[\frac{1}{y} + \frac{1}{b-y}\right]\hat{x}$; $d\vec{S} = -dy\,dz\,\hat{x}$



$\Phi = \frac{\mu_0 I}{2\pi}\int_a^{b-a}\left[\frac{1}{y} + \frac{1}{b-y}\right]dy \int_0^L dz = \frac{\mu_0 I L}{\pi}\ln\left[\frac{b-a}{a}\right]$

$= \frac{\mu_0 (1000)(100)}{\pi/4}\ln\left[\frac{1-0.02}{0.02}\right] = 155.67\ mWb.$