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c) $q(t) = \int_0^t i(t)\,dt = 0.125 \int_0^t e^{-25t}\,dt = -5\times10^{-3}\left[e^{-25t}\right]_0^t = 5\times10^{-3}\left(1 - e^{-25t}\right)\,C$

$t = 5\tau = 0.2\,s \Rightarrow q = 5\times10^{-3}\left(1 - e^{-25(0.2)}\right) = 4.9\cancel{6}6\,mC.$

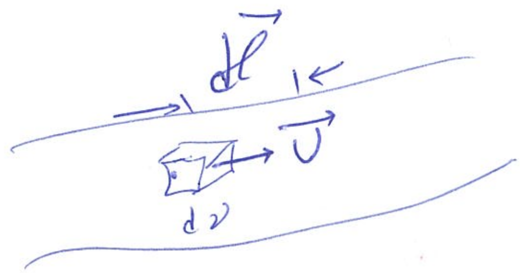
at $t = \infty \Rightarrow i(\infty) = 0$. No current.

Joule's law:

in a medium, charges moving with an ave. velocity $\vec{U}$ under $\vec{E}$.

ρ_v : The force experienced by the charge in a volume dv.

$$d\vec{F} = (\rho_v\, dv)\,\vec{E}$$



$d\vec{l} = \vec{U}\,dt$

The work done by the e-field:

$dW = d\vec{F}\cdot d\vec{l} = \rho_v\, \vec{U}\cdot\vec{E}\, dv\, dt \qquad \vec{J} = \rho_v \vec{U}$

$dW = \vec{J}\cdot\vec{E}\, dv\, dt$

Power $dP = \dfrac{dW}{dt} = \vec{J}\cdot\vec{E}\, dv = P\, dv$

Power density = Power per unit volume

$\boxed{P = \vec{J}\cdot\vec{E}}$ Point form (differential form) of Joule's law.

Power delivered per unit volume by the e-field

$P = \int_v p\, dv = \int_v \vec{J}\cdot\vec{E}\, dv$ → integral form of Joule's law.

In a conducting medium: The force ~~delivered~~ exerted to $\bar{e}$ by $\vec{E}$ are balanced by the loss in momentum during collisions.

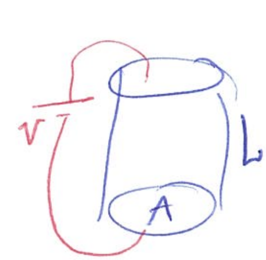
power delivered $\longrightarrow$ heat. $\Rightarrow$ Ohmic or Joule heating of resistors.

Power density.
$\Rightarrow$ $P =$ the time rate at which heat is being generated / volume?

For linear conductor:

$$\vec{J} = \sigma \vec{E} \Rightarrow \boxed{P = \sigma \vec{E} \cdot \vec{E} = \sigma E^2}$$

Total Power dissipation $\boxed{P = \int_v \sigma E^2 \, dv}$ [W].



$\Rightarrow$ V = Pot. diff. between two ends of a conductor of Length L & Uniform cross section A

$\Rightarrow$ Power density $P = \sigma \left[\frac{V}{L}\right]^2 \left[\frac{W}{m^3}\right]$

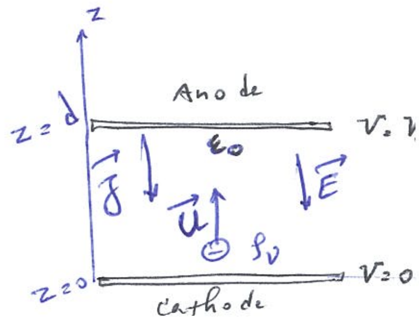
Total Power lost by the conductor: $P = \frac{\sigma V^2}{L^2}(LA) = \frac{\sigma A V^2}{L}$

used in electrical circuit theory (Power dissipated as heat by resistor) $\leftarrow$ $\boxed{P = \frac{V^2}{R}}$ [W], $R = \frac{L}{\sigma A}$

$P = \frac{R^2 I^2}{R} = I^2 R \longrightarrow$ the rate of heat dissipation varies as the square of I in linear conductor.

Steady current in a diode:

$\vec{E}$: from Higher Pot. to lower Pot.



$$\vec{E} = -\frac{dV}{dz}\hat{z}$$

cathode is heated to emit $\bar{e}$.

$\vec{U} = U\hat{z}$ velocity of an $\bar{e}$ at time t.

$e\vec{E} = m\frac{d\vec{U}}{dt}$

$$\vec{F} = m\vec{a} \Rightarrow e\frac{dV}{dz}\hat{z} = m\frac{dU}{dt}\hat{z} = m\frac{dU}{dz}\cdot\frac{dz}{dt}\hat{z} \Rightarrow m\frac{dU}{dz/U} = e\frac{dV}{dz}$$

$$\Rightarrow mU\frac{dU}{dz} = e\frac{dV}{dz} \Rightarrow \frac{d}{dz}\left[\frac{1}{2}mU^2 - eV\right] = 0$$

$$\int \Rightarrow \frac{1}{2}mU^2 = eV + C \quad ; \text{if } U_0 = 0 \text{ at } V = 0 \Rightarrow C = 0$$

$$\Rightarrow \boxed{\frac{1}{2}mU^2 = eV}$$ energy supplied by $\vec{E}$ is transformed into the kinetic energy of $\bar{e}$.

$$\Rightarrow U = \left[\frac{2eV}{m}\right]^{1/2}$$ velocity of $\bar{e}$ at any point.

⇒ we need distribution of V: $\frac{d^2V}{dz^2} = -\frac{\rho_v}{\epsilon_0}$ (1) ; $\rho_v = -Ne$, N: # of $\bar{e}$ / volume

& $\vec{J} = \rho_v\vec{U} = \rho_v U\hat{z} = J\hat{z}$

For steady current: $\nabla\cdot\vec{J} = 0 \Rightarrow \vec{J} = \rho_v\vec{U} = \text{const.} \rightarrow \boxed{U\uparrow \rightarrow \rho_v\downarrow}$

$$\Rightarrow \rho_v = \frac{J}{U} = \frac{J}{\sqrt{\frac{2eV}{m}}} = \frac{J/\sqrt{2e/m}}{\sqrt{V}} = \frac{k}{\sqrt{V}}.$$

(1) $\Rightarrow \int\frac{d^2V}{dz^2} = -\int\frac{k}{\epsilon_0\sqrt{V}} \rightarrow$

$$\frac{dV}{dz} = w \Rightarrow \frac{dw}{dz} = \frac{dw}{dV}\frac{dV}{dz} = w\frac{dw}{dV}$$

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$$\Rightarrow \int w\,dw = -\frac{k}{\epsilon_0}\int V^{1/2}\,dV \Rightarrow \frac{1}{2}w^2 = -\frac{k}{\epsilon_0}\,2\sqrt{V} + C$$

$$\Rightarrow w^2 = -4\frac{k}{\epsilon_0}\sqrt{V} + C$$

$$\left(\frac{dV}{dz}\right)^2 = -4\frac{k}{\epsilon_0}\sqrt{V} + C \quad ; \quad z=0, V=0 \ \& \ \frac{dV}{dz} = 0$$

$$\Rightarrow \boxed{C = 0}$$

$$\frac{dV}{V^{1/4}} = \sqrt{\frac{-4k}{\epsilon_0}}\,dz \Rightarrow \left(\frac{4}{3}\right)V^{3/4} = \sqrt{\frac{-4k}{\epsilon_0}}\,z + k_1$$

$$; z=0, V=0 \Rightarrow \boxed{k_1 = 0}$$

$$\Rightarrow \boxed{\frac{16}{9}V^{3/2} = -4\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,z^2} \quad (2)$$

; for $z=d$, $V=V_0$

nonlinear.

$$\boxed{J = \left(-\frac{4}{9}\right)\left(\frac{\epsilon_0}{d^2}\right)\sqrt{\frac{2e}{m}}\,(V_0)^{3/2}} \quad (3)$$

Child-Langmuir relation

$\vec{J}$ in $-\hat{z}$ direction. (e^- from cathode in $+\hat{z}$ direction $\Rightarrow \vec{J}$ in $-\hat{z}$ direction

substitute (3) in (2): $V = V_0\left[\frac{z}{d}\right]^{4/3}$.

$$\vec{E} = -\frac{dV}{dz}\hat{z} = -\frac{4V_0}{3d}\left[\frac{z}{d}\right]^{1/3}\hat{z}. \rightarrow E(z=0) = 0 \checkmark$$

conductor.

in practice it is small.

$$\nabla^2 V = -\frac{\rho_v}{\epsilon_0} \Rightarrow \frac{d^2V}{dz^2} = -\frac{\rho_v}{\epsilon_0} \Rightarrow \rho_v = -\frac{4\epsilon_0 V_0}{9d^{4/3}}z^{-2/3}.$$

Ex: The anode of a vacuum tube diode is at 1000 V, and cathode is grounded

the plates are 5 cm apart. a) $V(z) = ?$ b) $\vec{E} = ?$ c) $\vec{J} = ?$ d) $\rho_v = ?$

Pot. distribution

a) $V = V_0 \left(\frac{z}{d}\right)^{4/3} \Rightarrow V = 1000 \left(\frac{z}{0.05}\right)^{4/3} = 54.288\, z^{4/3}$ [kV].

b) $\vec{E} = -\frac{dV}{dz}\hat{z} = -\frac{4V_0}{3d^{4/3}} z^{1/3}\hat{z} = -72.384\, z^{1/3}\hat{z}$ $\left[\frac{kV}{m}\right]$.

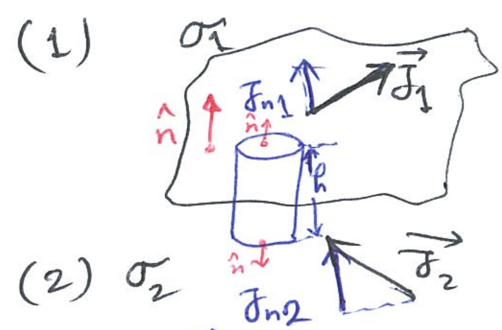
c) $\vec{J} = -\left(\frac{4}{9}\right)\left[\frac{10^{-9}}{36\pi(0.05)^2}\right]\left[\frac{2(1.6\times10^{-19})}{9.11\times10^{-31}}\right]^{1/2}(1000)^{3/2}\hat{z}$

$= -29.46\,\hat{z}$ $\left[\frac{A}{m^2}\right]$.

d) $\rho_v = -\frac{4\epsilon_0 V_0}{9d^{4/3}} z^{-2/3} = -\frac{4\times10^{-9}\times1000}{(9)(36\pi)(0.05)^{4/3}} z^{-2/3} = -213.34\, z^{-2/3}$ $\left[\frac{nC}{m^3}\right]$

Boundary conditions for current density:

How $\vec{J}$ changes when passing through an interface between media of σ_1 & σ_2?



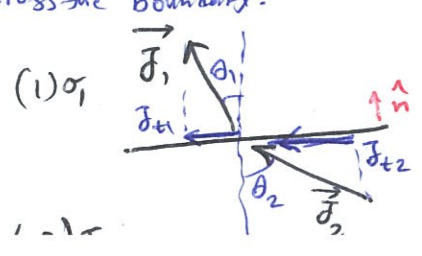
A closed surface (Pillbox); $h \to 0 \Rightarrow$ contribution from radial surface $\to 0$.

$\oint_s \vec{J}\cdot d\vec{S} = 0 \Rightarrow \hat{n}\cdot\vec{J}_1\,\Delta S - \hat{n}\cdot\vec{J}_2\,\Delta S = 0$

$\Rightarrow \hat{n}\cdot(\vec{J}_1 - \vec{J}_2) = 0 \Rightarrow \boxed{J_{n1} = J_{n2}}$ Normal component of $\vec{J}$ is continuous across the boundary.

We know: $E_{t1} = E_{t2} \Rightarrow \hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0$

$\vec{J} = \sigma\vec{E} \Rightarrow \hat{n}\times\left(\frac{\vec{J}_1}{\sigma_1} - \frac{\vec{J}_2}{\sigma_2}\right) = 0 \Rightarrow \boxed{\frac{J_{t1}}{J_{t2}} = \frac{\sigma_1}{\sigma_2}}$



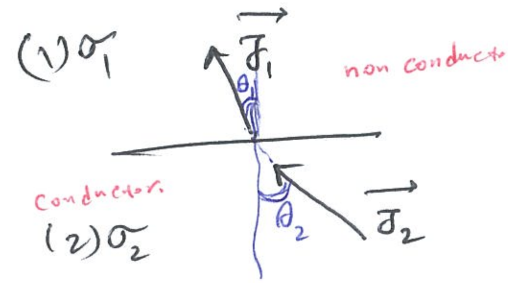
$$\Rightarrow \begin{cases} J_{n1} = J_{n2} \\ \dfrac{J_{t1}}{\sigma_1} = \dfrac{J_{t2}}{\sigma_2} \end{cases} \Rightarrow \boxed{\dfrac{J_{n1}\sigma_1}{J_{t1}} = \dfrac{J_{n2}\sigma_2}{J_{t2}}} \Rightarrow$$

$$\tan\theta_1 = \frac{J_{t1}}{J_{n1}}, \quad \tan\theta_2 = \frac{J_{t2}}{J_{n2}} \Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\sigma_1 \cancel{J_{t2}} \cancel{J_{n1}}}{\sigma_2 \cancel{J_{t2}} \cancel{J_{n1}}} = \frac{\sigma_1}{\sigma_2}}$$

check:

EX: medium 1: Poor conductor : σ_1 ; $\sigma_1 << \sigma_2$

medium 2: highly ~ : σ_2



if $\theta_2 < 90^\circ \Rightarrow \dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\sigma_1}{\sigma_2} \Rightarrow \tan\theta_1 = \dfrac{\sigma_1}{\sigma_2}\tan\theta_2 \Rightarrow \theta_1 \to 0$

$\Rightarrow \vec{J}_1$ & $\vec{E}_1$ are almost normal to the interface.

$\Rightarrow J_{1t} \simeq 0$, $\boxed{E_{1t} \simeq 0}$

$J_{n1} = J_{n2} \Rightarrow \sigma_1 E_{n1} = \sigma_2 E_{n2} \Rightarrow \boxed{E_{n2} = \dfrac{\sigma_1}{\sigma_2} E_{n1}} \to$ Very small. $E_{n2} \to 0$

$\boxed{D_{n2} = \dfrac{\sigma_1 \epsilon_2}{\sigma_2 \epsilon_1} D_{n1}}$

$\Rightarrow \vec{E}_2 \simeq 0 \Rightarrow$ E-field does not exist in a highly conducting medium. So there must exist a free surface charge density at the interface.

$\Rightarrow \rho_s = D_{n1} - D_{n2}$; $D_{n1} = \epsilon_1 E_{n1}$, $D_{n2} = \epsilon_2 E_{n2}$

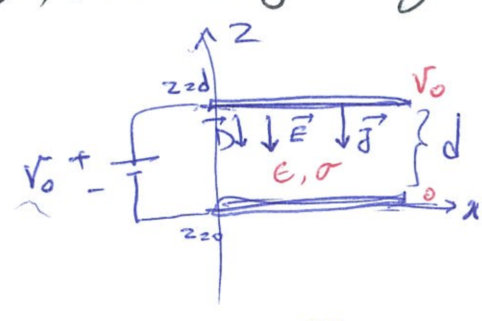
$= D_{n1}\left(1 - \dfrac{\sigma_1 \epsilon_2}{\sigma_2 \epsilon_1}\right) = E_{n1}\left[\dfrac{\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1}{\sigma_2}\right] = J_{n1}\left[\dfrac{\epsilon_1}{\sigma_1} - \dfrac{\epsilon_2}{\sigma_2}\right]$

EX.

Analogy between $\vec{D}$ & $\vec{J}$: In static conditions (time-invariant), both fields can be described by eqns of the same mathematical form:

$\vec{\nabla}\cdot\vec{J}=0$	$\vec{\nabla}\cdot\vec{D}=0$	$\cancel{\vec{J}=\sigma\vec{E},\ \vec{D}=\epsilon\vec{E}),\ \vec{\nabla}\times\vec{E}=0}$
$\vec{J}=\sigma\vec{E}$ For Linear medium	$\vec{D}=\epsilon\vec{E}$ For Linear medium:	$\Rightarrow \begin{cases}\vec{D}\to\vec{J}\\ \epsilon\to\sigma\end{cases}$
$\vec{\nabla}\times\vec{J}=0$	$\vec{\nabla}\times\vec{D}=0$	For a charge-free ~~medium~~ medium we first ~~solve~~
for 2 conducting media: $J_{n1}=J_{n2}$	for 2 dielectric media: $D_{n1}=D_{n2}$	$\Rightarrow$ assume that it is a dielectric $\to$ $\hat{D}$ then obtain the current density by substituting σ for ϵ.
$\dfrac{J_{t1}}{J_{t2}}=\dfrac{\sigma_1}{\sigma_2}$	$\dfrac{D_{t1}}{D_{t2}}=\dfrac{\epsilon_1}{\epsilon_2}$	

EX: Parallel-Plate capacitor $(V_0, A, d, \epsilon, \sigma)$. what is $\vec{J}$ using the analogy between $\vec{D}$ & $\vec{J}$.



$$\vec{E}=-\frac{V_0}{d}\hat{z}$$

$$\vec{D}=-\frac{\epsilon}{d}V_0\hat{z} \xrightarrow{\text{for a charge-free medium } (\sigma\to\epsilon)}$$

$$\Rightarrow \vec{J}=-\frac{\sigma}{d}V_0\hat{z} \Rightarrow I=\int_s \vec{J}\cdot d\vec{S}=\frac{\sigma A}{d}V_0=\frac{V_0}{R},\ R=\frac{d}{\sigma A}$$

conductance when C is known:

$$C=\frac{Q}{V_{ab}}=\frac{\int_s \rho_s ds}{-\int^a \vec{E}\cdot d\vec{l}}=\frac{\int_s \epsilon E_n ds}{-\int^a \vec{E}\cdot d\vec{l}}\ ;\ G=\frac{I}{V_{ab}}=\frac{\int \vec{J}\cdot d\vec{S}}{-\int^a \vec{E}\cdot d\vec{l}}=\frac{\int_s \sigma E_n ds}{-\int^a \vec{E}\cdot d\vec{l}} \Rightarrow \boxed{G=\frac{\sigma}{\epsilon}C}$$

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conductance $\quad$ capacitance

$$\boxed{G = \frac{\sigma}{\epsilon} C, \quad R = \frac{1}{G} = \frac{\epsilon}{\sigma C}.}$$

we can find the conductance → Resistance when we know $\underline{C}$ for any configuration

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The electromotive force:

- The $\int$ tangential comp. of $\vec{E}$ around any closed path vanishes: $\oint_C \vec{E}\cdot d\vec{\ell} = 0$ (electrostatic)

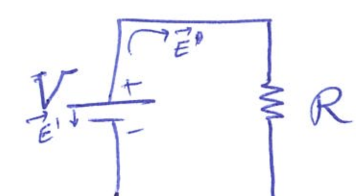
→ $I = \int_S \vec{J}\cdot d\vec{S} = \int_S \epsilon \vec{E}\cdot d\vec{S}$ → A purely electrostatic field cannot cause a current to circulate in a closed path (loop).

→ we need a source of energy in addition to e-field to maintain the steady current

↳ external source can be nonelectrical like $\underline{\text{chemical}}$ (battery), mechanical (direct-current generator), light-activated source (solar cell), temperature-sensitive device (thermo couple), nuclear (nuclear reactor).

⇒ Since these convert nonelect. energy to electrical → we consider them non conservative elements → they set up non conservative electric field $\vec{E}'$ in the loop.

⇒ $\vec{E}_{tot} = \vec{E} + \vec{E}'$



Total power associated with the loop:

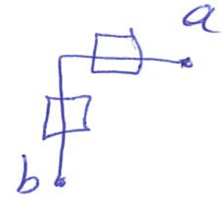
$$P = \int_V (\vec{E} + \vec{E}')\cdot \vec{J}\, dv \quad ; \text{ uniformly distributed steady current } \vec{J}\,dv = I\,d\vec{\ell}$$

$$= \oint I(\vec{E} + \vec{E}')\cdot d\vec{\ell} = I \oint_C \vec{E}'\cdot d\vec{\ell} = I\mathcal{E} \quad ; \quad \boxed{\mathcal{E} = \oint_C \vec{E}'\cdot d\vec{\ell}}$$

electromotive force (emf)

$\boxed{P = \mathcal{E} I}$ Power delivered to the loop

For a part of the loop, between a, b:



$$\int_a^b \frac{1}{\sigma}\vec{J}\cdot d\vec{l} = \int_a^b (\vec{E}+\vec{E}')\cdot d\vec{l}$$

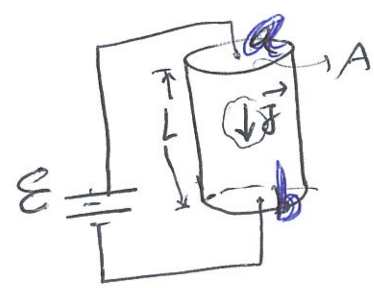
$$= \int_a^b \vec{E}\cdot d\vec{l} + \int_a^b \vec{E}'\cdot d\vec{l}$$

$$= -(V_b - V_a) + \mathcal{E}_{ab}$$ → emf of the sources between a & b.

if $\begin{cases} \mathcal{E}_{ab} = 0 \rightarrow \text{passive branch.} \\ \mathcal{E}_{ab} \neq 0 \Rightarrow \text{active } \text{〃} \text{ (seat of emf).} \end{cases}$

- In a cylindrical conductor of area A, length L between points a & b:

$$J = \frac{I}{A}, \quad \int_a^b \frac{1}{\sigma}\vec{J}\cdot d\vec{l} = \int \frac{I}{\sigma A}\, dl = \frac{IL}{\sigma A} = IR. \quad ; R = \frac{L}{\sigma A}.$$



$$\Rightarrow IR = -[V_b - V_a] + \mathcal{E}_{ab}$$

if $\mathcal{E}_{ab} = 0 \Rightarrow \boxed{V_a - V_b = IR}$ voltage drop across the resistor.

if $V_a > V_b \rightarrow \underline{I}$ is positive. : current enters $\underline{a}$ & leaves at $\underline{b}$.

If R = total ressistance of the circuit, if a & b are the same:

$$\mathcal{E} = IR.$$

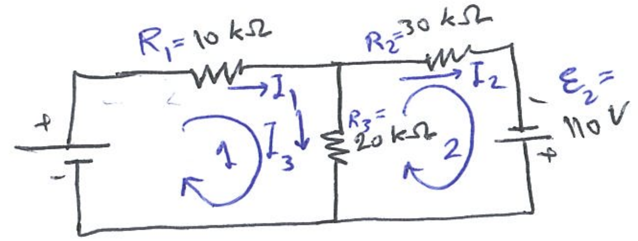
$$\mathcal{E} = \sum_{k=1}^{m} \mathcal{E}_k = \sum_{j=1}^{n} IR_j \rightarrow$$ Kirchhoff's voltage law.

algebraic sum of the emfs in any closed loop = sum of the voltage drops in the same loop.

EX: I of each element & Total Power supplied by the sources?

→ direction of currents: arbitrary.

$\mathcal{E}_1 = 110\,V$



$\sum I_i = 0 \Rightarrow I_1 = I_2 + I_3$

or $I_3 = I_1 - I_2$

loop ①: $R_1 I_1 + R_3 I_3 = \mathcal{E}_1$

loop ②: $R_2 I_2 - R_3 I_3 = \mathcal{E}_2$

$$\Rightarrow \begin{cases} I_1 + 2I_3 = 0.011 \\ 3I_2 - 2I_3 = 0.011 \end{cases} \Rightarrow \begin{cases} 3I_1 - 2I_2 = 0.011 \\ -2I_1 + 5I_2 = 0.011 \end{cases}$$

$$\Rightarrow \begin{cases} I_1 = 7\,mA \\ I_2 = 5\,mA \\ I_3 = 2\,mA \end{cases}$$

Total power delivered to the circuit:

$$P_s = \sum_i \mathcal{E}_i I_i = \mathcal{E}_1 I_1 + \mathcal{E}_2 I_2 = (110)(7\times10^{-3}) + (110)(5\times10^{-3}) = 1.32\,W.$$

Total power dessipated by resistors as heat:

$$P_d = \sum_i R_i I_i^2 = R_1 I_1^2 + I_2^2 R_2 + I_3^2 R_3 = (10\times10^3)(0.007)^2 + (30\times10^3)(0.005)^2 + (20\times10^3)(0.002)^2 = 1.32\,W. ✓$$

Problems: 4.1, 4.4, 4.7, 4.12, 4.18, 4.26, 4.28, 4.33, 4.41