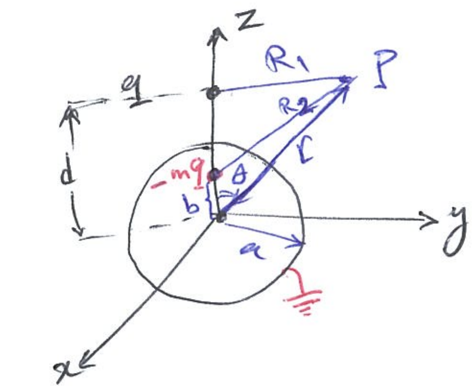
EX: A point charge q is placed at a distance d from the center of a grounded conducting shere of radius a. $\rho_s = ?$



$$V_p = \frac{q}{4\pi\epsilon}\left[\frac{1}{R_1} - \frac{m}{R_2}\right]$$

$$\begin{cases} R_1 = [r^2 + d^2 - 2rd\cos\theta]^{1/2} \\ R_2 = [r^2 + b^2 - 2rb\cos\theta]^{1/2} \end{cases}$$

$$V(r=a) = 0 \Rightarrow \frac{1}{[a^2+d^2-2ad\cos\theta]^{1/2}} = \frac{m}{[a^2+b^2-2ab\cos\theta]^{1/2}}$$

$$\Rightarrow m^2(a^2+d^2-2ad\cos\theta) = a^2+b^2-2ab\cos\theta$$

$$\Rightarrow \begin{cases} m^2(a^2+d^2) = a^2+b^2 \\ 2ad\,m^2 = 2ab \end{cases} \Rightarrow m^2 = \frac{b}{d}$$

$$\frac{b}{d}(a^2+d^2) = a^2+b^2$$

$$b^2 - \left(\frac{a^2+d^2}{d}\right)b + a^2 = 0$$

$$b = \frac{a^2}{d}$$

$$m^2 = \frac{a^2}{d^2} \Rightarrow m = \pm\frac{a}{d}$$

$m \le 1 \Rightarrow$ at $b = a \Rightarrow V = 0 \Rightarrow$ $m = \frac{a}{d}$

$$\vec{D}\cdot\hat{r} = \rho_s \Rightarrow -\epsilon\vec{\nabla}V\cdot\hat{r} = -\epsilon\left.\frac{\partial V}{\partial r}\right|_{r=a} = -\frac{q}{4\pi a}\left[\frac{d^2-a^2}{(d^2+a^2-2ad\cos\theta)^{3/2}}\right] = \rho_s$$

$$V(r,\theta,\varphi) = \frac{q}{4\pi\epsilon}\left[\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{a}{d(r^2+b^2-2rb\cos\theta)^{1/2}}\right]$$

chapter 4 Steady E_ currents

→ Discuss the motion of charges in a conducting medium when an e-field is maintained within the conductor.

current: The rate at which the charge is transported past a given point in a conducting medium.

$$[\text{Ampere}] \quad i = \frac{dq}{dt}$$

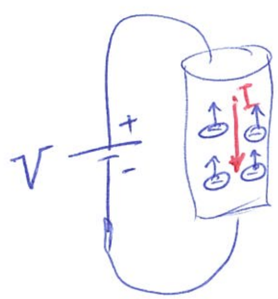
→ the amount of charge that flows past some given point in time dt.

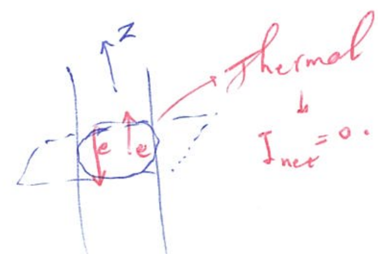
Steady current (direct current): the currents that are constant in time. I.

- conduction currents: flow of free electrons in metals (copper, gold, silver, ...).

If we put a metal in an e-field the ~~charges~~ electrons will move temporarily ($t \ll 1s$) and stop. → To maintain a steady current within a conductor, a continuous supply of electrons at one end & their removal at the other is necessary.

⇒ $\bar{e}$ move within wire & do collisions with ions $\simeq 10^{14}$ $\frac{\text{collisions}}{\text{second}}$.





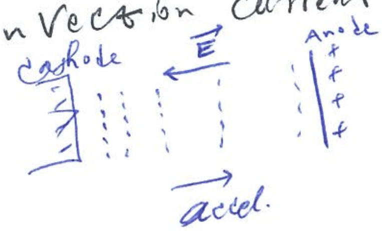
($\bar{e}$ → it stops or changes its direction.

E-field must work again to change the $\vec{v}_e$. → drift velocity ≪ random velocity.

⇒ The net drift of $\bar{e}$s in z-direction constitutes a current through the conductor. direction of I = direction of e-field.

→ "No source", "No sink" in a wire. (conservation of charges).

- Convection current: motion of charged particles in free space (vacuum).



$v_{\bar{e}}$ increases → no collision. → charge density decreases.

so: → convection current is not electrostatically neutral & its electrostatic charge must be taken into account.

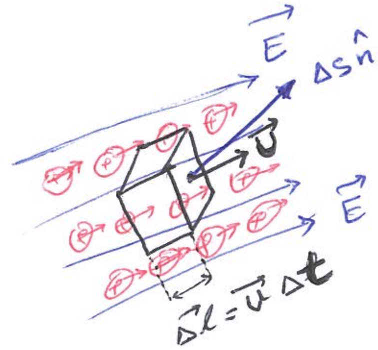
→ does not need conductor to maintain charge flow, nor obeys Ohm's law.

Convection current density:

$\vec{U}$ = drift velocity
$\vec{U}$ = average velocities of charges → steady current.

$$dq = \rho_v \Delta v = \rho_v \vec{\Delta S} \cdot \vec{dl}$$

$$\Delta I = \frac{dq}{dt} = \rho_v \vec{\Delta S} \cdot \left(\frac{\vec{dl}}{dt}\right) = \rho_v \vec{\Delta S} \cdot \vec{U}$$



$$\Rightarrow \Delta I = \rho_v \vec{U} \cdot \vec{\Delta S} \equiv \vec{J} \cdot \vec{\Delta S}$$

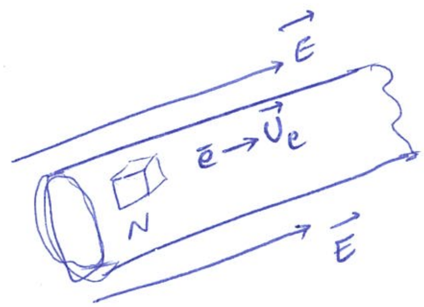
current density $\boxed{\vec{J} = \rho_v \vec{U}}$ convection current density.

The current passing through a surface S: $I = \int_S \vec{J} \cdot d\vec{S}$

For both charges (+) & (−) the current is in the same direction $\Rightarrow \vec{J} = \rho_{v_+} \vec{U}_+ + \rho_{v_-} \vec{U}_-$

Conduction Current density:

an $\bar{e}$, $\vec{U}_e$, m_e, τ = mean time per collision.



momentum lost, in time τ for an $\bar{e}$:

$$m_e \vec{U}_e$$

Average rate at which an $\bar{e}$ loses momentum $= \dfrac{m_e \vec{U}_e}{\tau}$

The rate at which $\bar{e}$ gains momentum from e-force $= -e\vec{E}$

} → under steady state conditions, the rate of loss of momentum = rate of gain

$$\frac{m_e \vec{U}_e}{\tau} = -e\vec{E} \Rightarrow \vec{U}_e = -\frac{e\tau\vec{E}}{m_e} = -\mu_e \vec{E}$$

↳ $\mu_e = \dfrac{e\tau}{m_e}$: $\bar{e}$ mobility. const. of proportionality

↙ drift velocity of an $\bar{e}$ $\propto \vec{E}$.

N = # of $\bar{e}$ per unit volume $\Rightarrow \rho_{v_-} = -Ne$ (electron charge density) conduction current density $\vec{J} = \rho_{v} \vec{U_e}$

microscopic equivalent of ohm's law. $\leftarrow$ $\vec{J} = (-Ne)(-u_e \vec{E}) = N e u_e \vec{E} = \sigma \vec{E}$

$\sigma = N e u_e$ = conductivity of the medium. $\frac{\text{Siemens}}{m}$ $[S/m]$.

$\rightarrow$ $\vec{J} \propto \vec{E}$ in a conducting medium.

$\vec{J} = \sigma \vec{E}$ $\rightarrow$ const. of proportionality.

$\rightarrow$ For linear medium: $\vec{J}$ & $\vec{E}$ are in the same direction.

ohm's law: it's valid if $R \neq R(V, I)$: $\underline{R}$ indep. of V & I. $\rightarrow$ for circuits

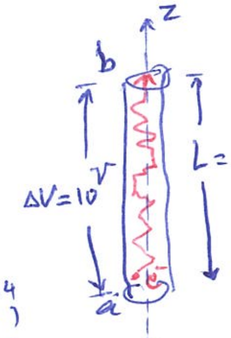
Conducting material obeys ohm's law if $\sigma \neq \sigma(E)$.
$\downarrow$
σ indep. of e-field.

Linear or ohmic media: $\vec{J} = \sigma \vec{E}$.

resistivity: $\rho = \frac{1}{\sigma}$ $(\Omega.m)$.

material	ρ $(\Omega.m)$
metals	
Aluminum	2.83×10^{-8}
copper	1.72×10^{-8}
Gold	2.44×10^{-8}
Iron	8.9×10^{-8}
silver	1.47×10^{-8}
Tungsten	5.51×10^{-8}
semi conductors	
carbon	3.5×10^{-5}
Germanim	0.42
silicon	2.6×10^{3}
insulators	
Amber	5×10^{14}
Quartz	7.5×10^{17}
Glass	$10^{10} - 10^{14}$

Ex: Calculate the drift velocity of free $\bar{e}$ in 2^m wire of copper with $\underline{10}^V$ pot. diff. & $\tau = 2.7 \times 10^{-14}$ s.



$\vec{E} = -\vec{\nabla} V = -\left(\frac{\Delta V}{L}\right) \hat{z}$

$= -\frac{10}{2} \hat{z} = -5 \hat{z}$ $[V/m]$.

$\bar{e}$ mobility: $u_e = \frac{e\tau}{m_e} = \frac{(1.6 \times 10^{-19})(2.7 \times 10^{-14})}{9.1 \times 10^{-31}}$ m/s

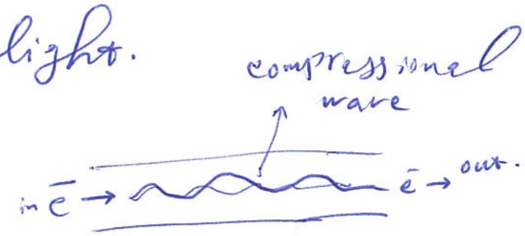
$= 4.747 \times 10^{-3}$

$\vec{U_e} = -u_e \vec{E} = -(4.747 \times 10^{-3})(-5 \hat{z}) = 23.74 \times 10^{-3} \hat{z}$ $[m/s]$

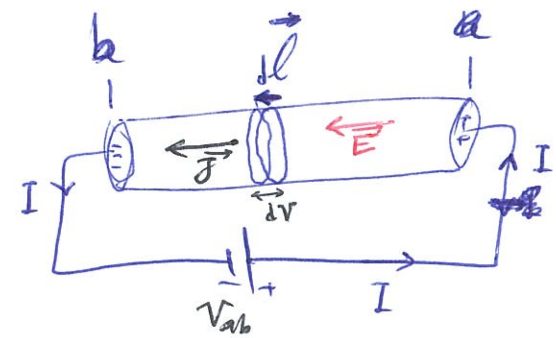
[illegible]

* currents "travels" through the wire at speed of light.

One $\bar{e}$ enters at one end, it pushes the neighboring $\bar{e}$ by means of e-field → creates a compressional wave within the wire. The compressional wave travels with c & ejects $\bar{e}$s out of wire at the far end almost instantaneously.



Resistance of a conductor:



$I \propto \Delta V$

$\Rightarrow \Delta V = RI$ (R: resistance.)

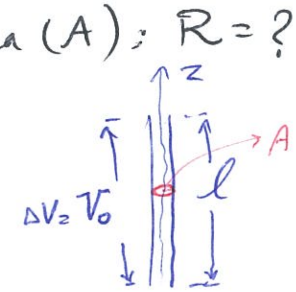
$$dR = \frac{dV}{I} = \frac{-\vec{E}\cdot d\vec{l}}{\int_s \vec{J}\cdot d\vec{s}}$$

$V_a > V_b$.

$$\Rightarrow R = \frac{-\int_b^a \vec{E}\cdot d\vec{l}}{\int_s \vec{J}\cdot d\vec{s}} = \frac{V_{ab}}{I} \Rightarrow \boxed{R = \frac{V_{ab}}{I}}$$

calculating R is not always easy!

EX: A copper wire (l), pot. diff. (V_0), cross-sectional area (A); $R = ?$



$R = \frac{\Delta V}{I}$

$$\vec{E} = -\vec{\nabla} V = -\frac{V_0}{l}\hat{z}$$

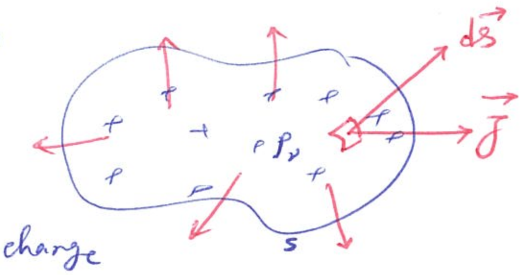
$$\vec{J} = \sigma\vec{E} = -\frac{\sigma V_0}{l}\hat{z} \Rightarrow I = \int_s \vec{J}\cdot d\vec{s} = \frac{\sigma V_0}{l}\int_s ds = \frac{\sigma V_0 A}{l}$$

$$R = \frac{V_0}{I} = \frac{l}{\sigma A} = \boxed{\frac{\rho l}{A}}$$

The eqn of continuity:

A conducting region, closed surface S, ρ_v = vol. charge density.

$i(t) = \oint_S \vec{J} \cdot d\vec{S}$ → Total I crossing the closed surface $\underline{S}$ outward direction.



current = flow of charge per second. ⇒ decrease the charge concentration.

$i(t) = -\dfrac{dQ}{dt}$, Q = total charge enclosed by $\underline{S}$ at any time $\underline{t}$.

$= \int_v \rho_v \, dv$

$$\Rightarrow \boxed{\oint_S \vec{J} \cdot d\vec{S} = -\frac{d}{dt}\int_v \rho_v \, dv}$$

} integral form of the eqn. of continuity. ≡ conservation of charge.

⇒ no charge creation, nor destroyed, but transported.

divergence theorem: $\oint_S \vec{J} \cdot d\vec{S} = \int_v \vec{\nabla} \cdot \vec{J} \, dv$ ⇒ $\int_v \left(\vec{\nabla} \cdot \vec{J} + \dfrac{\partial \rho_v}{\partial t} \right) dv = 0$

stationary volume: $\dfrac{d}{dt}\int_v \rho_v \, dv = \int_v \dfrac{\partial \rho_v}{\partial t} \, dv$ (v ↘ arbitrary)

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = 0}$$

diff. form of contin.

$\vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$ → changing charge density ρ_v is the source of vol. current density $\vec{J}$.

for a conducting medium to sustain a steady current: $\dfrac{\partial \rho_v}{\partial t} = 0$

⇒ $\oint_S \vec{J} \cdot d\vec{S} = 0$ ⇒ $\boxed{\vec{\nabla} \cdot \vec{J} = 0}$ → the net steady current through any closed surface is zero

↳ for a steady $\underline{I}$ through a conducting medium, $\vec{J}$ is continuous or solenoidal

→ shrink $\underline{S}$ to a point: $\boxed{\sum I = 0}$ Kirchhoff's Current Law.

→ Algebraic sum of the currents at a point (junction or node) $= 0$.

6#

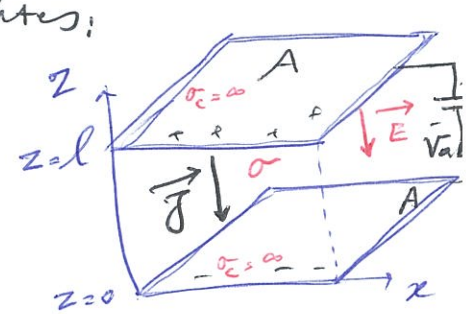
$\vec{J} = \sigma \vec{E} \Rightarrow \vec{\nabla}\cdot(\sigma\vec{E}) = 0 \Rightarrow \sigma\vec{\nabla}\cdot\vec{E} + \vec{E}\cdot\vec{\nabla}\sigma = 0$

For homogeneous medium $\vec{\nabla}\sigma = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{E} = 0 \; ; \; \vec{E} = -\vec{\nabla}V \Rightarrow \boxed{\nabla^2 V = 0}$

→ Pot. distribution within a conducting medium satisfies LaPlace's eqn, if medium is homogeneous & the current distribution is time invariant.

EX: Two ∞ conducting parallel plate, distance ℓ, V_{ab}, σ = conductivity of medium between two plates (homog.) $\xrightarrow{\text{find}}$ $\underline{R}$ of region between plates;



$\nabla^2 V = 0$ for the region, homog., σ.

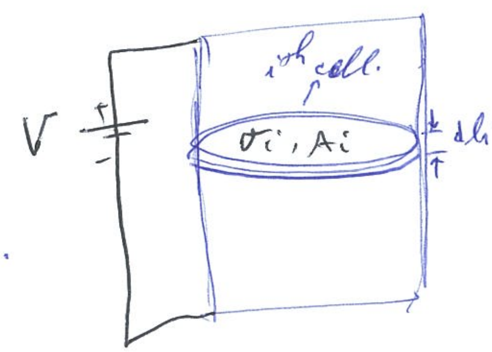
$$\frac{d^2V}{dz^2} = 0 \Rightarrow V = az + b.$$

B.C. $\begin{cases} z=0,\ V=0 \Rightarrow b=0. \\ z=\ell,\ V=V_{ab} \Rightarrow a = \dfrac{V_{ab}}{\ell}. \end{cases}$ $\Rightarrow \boxed{V = \dfrac{V_{ab}}{\ell} z}$

$$\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial z}\hat{z} = -\frac{V_{ab}}{\ell}\hat{z}$$

$$\vec{J} = \sigma\vec{E} = -\frac{\sigma V_{ab}}{\ell}\hat{z} \quad , \; I = \int_s \vec{J}\cdot d\vec{S} = \frac{\sigma A V_{ab}}{\ell} \rightarrow R = \frac{V_{ab}}{I} = \frac{\ell}{\sigma A} = \frac{\rho\ell}{A}.$$

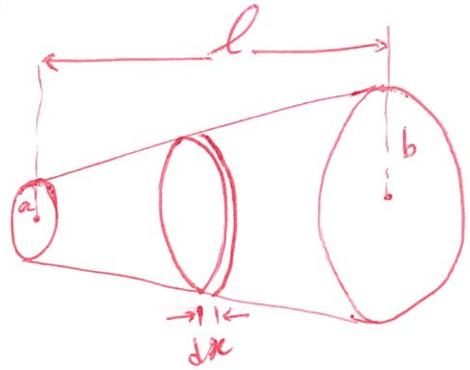
For nonhomogeneous conducting medium:

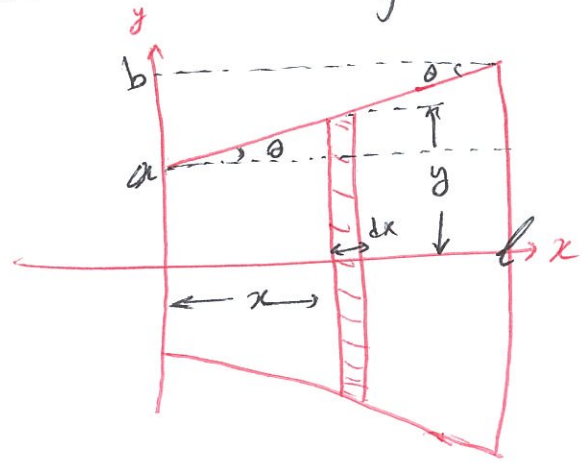


$R_i = \dfrac{dl_i}{\sigma_i A_i}$ each small cell has const. conductivity σ_i.

$$R = \sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \frac{dl_i}{\sigma_i A_i} \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{dl_i}{\sigma_i A_i} = \int_c \frac{dl}{\sigma A}.$$

Example: A Frustum is made with a material with conductivity of $\sigma\left(\frac{S}{m}\right)$ and the base radiai a and b and the length of l. Assume the current density is constant. Now we apply silver coating on both bases of the cone. Find the electric resistance between two bases. What this resistance will be if $a = b$.





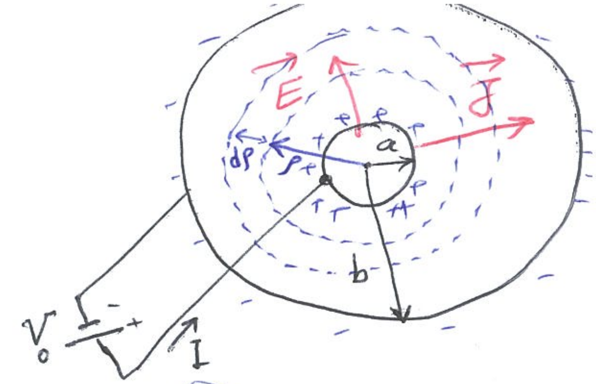
A short cyliner is our small element of medium with thickness dx and radius y.

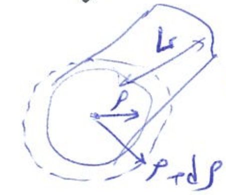
$$dR = \frac{dx}{\sigma(\pi y^2)} \quad ; y = a + x\tan\theta = a + \left(\frac{b-a}{l}\right)x$$

$$\Rightarrow R = \int_0^l \frac{dx}{\sigma\pi y^2} = \int_0^l \frac{dx}{\sigma\pi\left[a + \left(\frac{b-a}{l}\right)x\right]^2}$$

$$= \frac{1}{\sigma\pi}\left\{\frac{-1}{\left(\frac{b-a}{l}\right)\left[\left(\frac{b-a}{l}\right)x + a\right]}\right\}_0^l = \frac{l}{\sigma\pi ab}$$

EX: Between 2 concentric cylindrical conductors of radii $\underline{a}$, $\underline{b}$ is filled with $\sigma = \frac{m}{\rho} + k$. V_0 = pot. diff. between $\underline{a}$ & $\underline{b}$. ; L = length of conductors. $R_{material} = ?$, $\vec{J} = ?$, $\vec{E} = ?$





cylindrical cross. section for ρ = $A_i = 2\pi\rho L$.

method 1:

$$R = \int_c \frac{dl}{\sigma A} = \int_a^b \frac{d\rho}{\left(\frac{m}{\rho} + k\right) 2\pi\rho L}$$

$$R = \int_a^b \frac{d\rho}{(m + k\rho) 2\pi L} = \frac{1}{2\pi L} \cdot \frac{1}{k} \ln(m + k\rho) \Big|_a^b$$

$$= \frac{1}{2\pi L k} \left(\ln \frac{m + kb}{m + ka} \right) = \boxed{\frac{M}{2\pi L k}}$$

$$\ln \frac{m + kb}{m + ka} \to M$$

method 2:

$$R = \frac{V_{ab}}{I} = \frac{V_0}{I}$$

$$\vec{J} = \frac{I}{2\pi\rho L} \hat{\rho} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \frac{I}{2\pi\rho L \left(\frac{m}{\rho} + k\right)} \hat{\rho} = \frac{I}{2\pi L (m + k\rho)} \hat{\rho}$$

$$V_{ab} = V_0 = -\int_b^a \vec{E} \cdot d\vec{l} = -\int_b^a \frac{I \, d\rho}{2\pi L (m + k\rho)} = \frac{I}{2\pi L k} \ln \left[\frac{m + kb}{m + ka} \right] = \frac{I M}{2\pi L k}$$

$$\Rightarrow \boxed{R = \frac{V_0}{I} = \frac{M}{2\pi L k}}$$

$$\Rightarrow \boxed{I = \frac{2\pi L k}{M} V_0} \quad ; \quad \vec{E} = \frac{I}{2\pi L (m + k\rho)} \hat{\rho} = \frac{2\pi L k V_0}{M 2\pi L (m + k\rho)} \hat{\rho} = \boxed{\frac{k V_0}{(m + k\rho) M} \hat{\rho}}$$

$$\boxed{\vec{J} = \sigma \vec{E} = \frac{k V_0}{M \rho} \hat{\rho}}$$

Relaxation Time: For an isolated, linear, homogeneous & isotropic medium with permittivity ϵ & conducting σ having an excess charge density of $\rho_v \Rightarrow$ charges go to boundaries:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = 0$$

eq. of cont. must be satisfied during this migration of charges.

$$\vec{J} = \sigma \vec{E}$$

$$\sigma \vec{\nabla} \cdot \vec{E} + \frac{\partial \rho_v}{\partial t} = 0 \quad ; \vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon} \rightarrow \text{Gauss's law}$$

equilibrium is reached exponentially.

$$\Rightarrow \frac{\sigma \rho_v}{\epsilon} + \frac{\partial \rho_v}{\partial t} = 0 \Rightarrow \boxed{\frac{\partial \rho_v}{\partial t} + \left(\frac{\sigma}{\epsilon}\right) \rho_v = 0} \Rightarrow \rho_v = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right)t} = \rho_0 e^{-\frac{t}{\tau}}$$

ρ_0: ρ_v at $t = 0$.

$\rightarrow t \rightarrow \infty$ for $\rho_v \rightarrow 0$.

$$\left[\frac{\epsilon}{\sigma}\right] = \text{time} \Rightarrow \boxed{\tau = \frac{\epsilon}{\sigma} = \text{relaxation time.}}$$

$\equiv$ A measure of how fast a conductor reaches electrostatic equilibrium.

if $\begin{cases} t = \tau \Rightarrow \rho_v = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right)\frac{\epsilon}{\sigma}} = \rho_0 e^{-1} = \frac{\rho_0}{e} = \frac{\rho_0}{2.7} = 36.8\% \rho_0. \\ t = 5\tau \rightarrow \rho_v < 1\% \rho_0 \Rightarrow \text{electrostatic equilib.} \end{cases}$

For copper $\sigma = 5.8 \times 10^7$ S/m, $\epsilon \simeq \epsilon_0 \Rightarrow \tau = 1.52 \times 10^{-19}$ s.

Pure water, $\tau = 40 \times 10^{-9}$ s

Amber, $\tau = 70$ minutes.

EX: A Q_0 is placed within an isolated conductor. I through a closed S (bounding the charge) is

$i(t) = 0.125 e^{-25t}$ A) τ B) Q_0 C) q transported through S in time $t = 5\tau$.

A) $i(t) = i_0 e^{-\frac{t}{\tau}} \Rightarrow \tau = \frac{1}{25} = 0.04$ s.

B) $Q_0 = \int_0^\infty i \, dt = \int_0^\infty (0.125) e^{-25t} dt = \frac{0.125}{-25} e^{-25t} \Big|_0^\infty = -5 \times 10^{-3} [0 - 1] = 5 \times 10^{-3}$ C.