→ Two dielectric interface: No free surface charge ⇒ $\rho_s = 0$ → $\vec{D}$ is continuous.

$$D_{n1} = D_{n2} \Rightarrow \epsilon_1 E_{n1} = \epsilon_2 E_{n2}$$

→ medium 2 is a conductor: $\vec{D_2} = 0$ under static conditions;

To have $\vec{D_1} \neq 0$ in medium 1, we must have a free surface charge density on the conductors surface:

$$\begin{cases} \hat{n} \cdot \vec{D_1} = D_{n1} = \rho_s \\ \epsilon_1 E_{n1} = \rho_s \end{cases} \Rightarrow$$

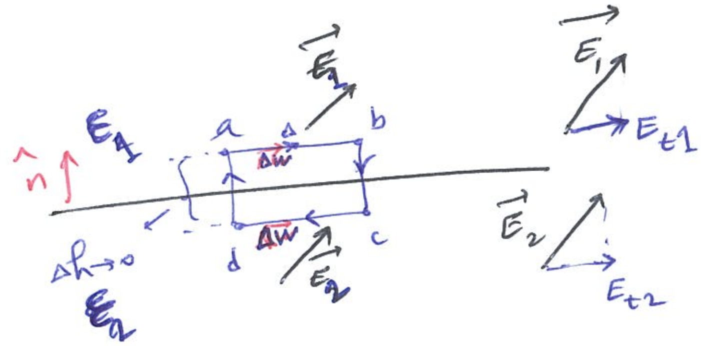
D_n in a dielectric medium just above the surface of a conductor = ρ_s on the conductor.

The Tangential component of $\vec{E}$:

E-field is conservative in nature:

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\Delta h \to 0 \Rightarrow \oint \int_d^a \vec{E} \cdot d\vec{l} = \int_b^c \vec{E} \cdot d\vec{l} = 0$$



$$\oint \vec{E} \cdot d\vec{l} = \vec{E_1} \cdot \vec{\Delta W} - \vec{E_2} \cdot \vec{\Delta W} = 0$$

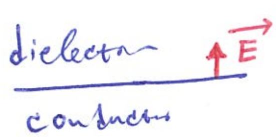
$$\Rightarrow (\vec{E_1} - \vec{E_2}) \cdot \vec{\Delta W} = 0 \quad ; \text{ if } \vec{\Delta W} = \Delta W \hat{t}$$

$$\to \hat{t} \cdot (\vec{E_1} - \vec{E_2}) = 0 \Rightarrow \boxed{E_{t1} = E_{t2}}$$

→ the tangential components of e-field intensity are always continuous at the interface.

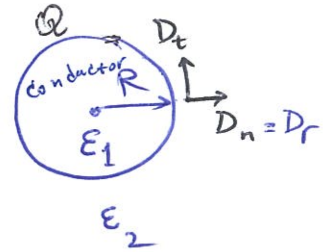
$$\boxed{\hat{n} \times (\vec{E_1} - \vec{E_2}) = 0}$$

⇒ if $\begin{cases} \text{medium 1 = dielectric} \\ \text{medium 2 = conductor} \end{cases} \Rightarrow E_{t1} = E_{t2} = 0 \Rightarrow$ The electrostatic field just above a conductor is always $\perp$ to the surface of conductor.



EX: A metallic sphere of radius R with charge Q on its surface. what is $\vec{E}$ just above the surface of sphere?

$$\rho_s = \frac{Q}{4\pi R^2} \quad ; \quad \vec{D}_t = 0 \quad , \quad \vec{D}_n = D_r \hat{r}$$

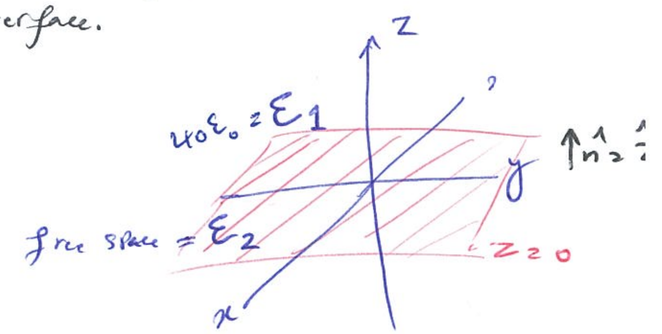


$$\hat{n} \cdot \vec{D}_1 = D_{n1} = \rho_s \Rightarrow \boxed{D_r = \frac{Q}{4\pi R^2}}$$

$$\Rightarrow E_r = \frac{D_r}{\epsilon_2} = \frac{Q}{4\pi \epsilon_2 R^2} \longrightarrow \text{check by Gauss's law.}$$

EX: $z=0$ plane marks the boundary between free space & a dielectric medium with dielec const. of 40. $\vec{E}$ next to the interface in free space $\Rightarrow \vec{E}_{free} = 13\hat{x} + 40\hat{y} + 50\hat{z}$

Determine the E-field on the other side of interface.



$$\vec{E}_2 = 13\hat{x} + 40\hat{y} + 50\hat{z}$$

$$E_{t1} = E_{t2} \Rightarrow \begin{cases} E_{x1} = E_{x2} = 13 \\ E_{y1} = E_{y2} = 40 \end{cases}$$

$$D_{n1} = D_{n2} \Rightarrow \epsilon_1 E_{z1} = \epsilon_2 E_{z2} \quad ; \begin{cases} \epsilon_1 = 40\epsilon_0 \\ \epsilon_2 = \epsilon_0 \end{cases}$$

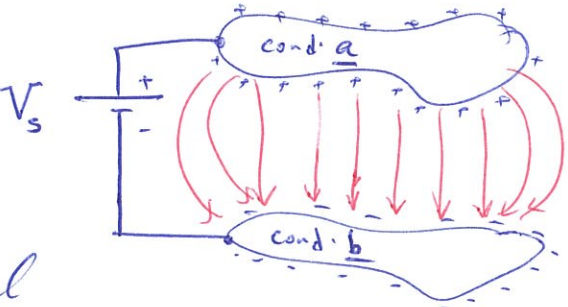
$$\Rightarrow E_{z1} = \frac{\epsilon_2 E_{z2}}{\epsilon_1} = \frac{\epsilon_0 (50)}{40\epsilon_0} = 1.25$$

$$\Rightarrow \vec{E}_1 = 13\hat{x} + 40\hat{y} + 1.25\hat{z} \ [V/m].$$

Capacitors & Capacitance

: Two any shape insulated conductors adjacent to each other form a **capacitor**.

Charging the capacitor: Transfer charges from one conductor to another using an external source of energy.



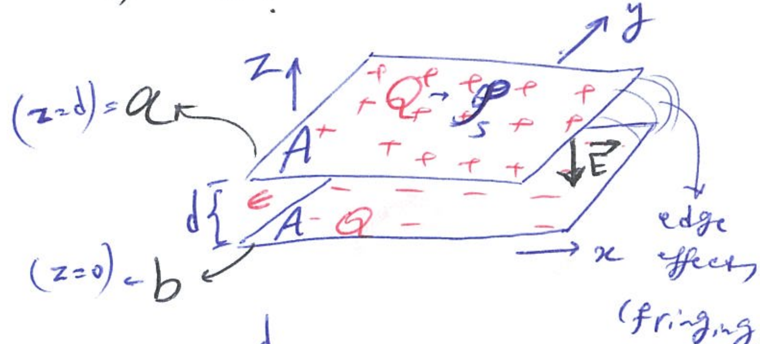
→ This transfer stablishes an e-field in the dielectric medium and thereby a potential difference between the conductors.

$$\Rightarrow V_{ab} \propto Q_{transfered} \Rightarrow Q_a = C V_{ab}$$

capacitance → $$\overset{[F]}{C} = \frac{Q_a\,[C]}{V_{ab}\,[V]}.$$

Ex: Parallel-Plate capacitor (A, Q, d); $C = ?$, $W = ?$ stored energy in the medium.

assume $d \ll A$:



$$\vec{E} = \frac{-\rho_s}{\epsilon}\hat{z} \quad ; \quad \rho_s = \frac{Q}{A}$$

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_0^d \frac{-\rho_s}{\epsilon}\hat{z}\cdot dz(\hat{z}) = \int_0^d \frac{\rho_s\, dz}{\epsilon}$$

$$= \frac{\rho_s}{\epsilon}\int_0^d dz = \frac{\rho_s d}{\epsilon} = \frac{Q d}{\epsilon A}$$

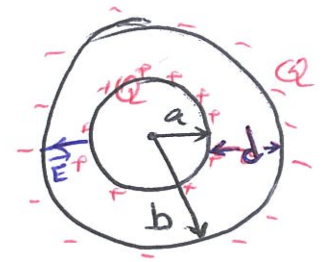
$$\boxed{C = \frac{Q}{V_{ab}} = \frac{\epsilon A}{d}}$$

$$W = \frac{1}{2}\int_v \epsilon E^2 dv = \frac{\epsilon}{2}\int\left(\frac{\rho_s}{\epsilon}\right)^2 dv = \frac{1}{2}\frac{\rho_s^2}{\epsilon} A d$$

$$= \boxed{\frac{1}{2}\frac{d}{\epsilon A}Q^2} = \boxed{\frac{1}{2C}Q^2} = \boxed{\frac{1}{2}C V_{ab}^2}$$

Basic circuit equations for energy stored in a capacitor.

Ex: spherical capacitors: Two concentric metallic spheres of radii a & b. Find capacitance? capacitance of an isolated sphere? C of earth ($r = 6.5\times10^{6}$ m)? for ($d << r$)?



$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

$$V_{ab} = -\frac{Q}{4\pi\epsilon}\int_b^a \frac{1}{r^2}\,dr = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$\boxed{C = \frac{Q}{V_{ab}} = \frac{4\pi\epsilon ab}{b-a}}$$

- isolated sphere: $b \to \infty$: $C = 4\pi\epsilon a$

- earth $\epsilon = \epsilon_0 \Rightarrow C_{earth} = \dfrac{6.5\times10^{6}}{9\times10^{9}} = 0.722\times10^{-3}\,F \Rightarrow C_{earth} \simeq 722\,\mu F$

- if $d << a \Rightarrow ab \simeq a^2 \Rightarrow C = \dfrac{4\pi\epsilon a^2}{b-a} = \dfrac{\epsilon A}{d}$. ; $A = 4\pi a^2$ surface area of inner sphere.

Ex: The region between two concentric spherical shells is filled with two different dielectrics. Find capacitance of the system.

$\vec{E} = E\hat{r}$

The tangential components of $\vec{E}$ is continuous at the boundary between two media:

$$E_{r1} = E_{r2} \Rightarrow \frac{D_{r1}}{\epsilon_1} = \frac{D_{r2}}{\epsilon_2} \Rightarrow D_{r2} = \frac{\epsilon_2}{\epsilon_1} D_{r1}$$

half sphere

Gauss's law $a \le r \le b$: $\oint \vec{D}\cdot d\vec{S} = Q \Rightarrow D_{r1}(2\pi r^2) + D_{r2}(2\pi r^2) = Q$

$$\Rightarrow \boxed{D_{r_1} + D_{r_2} = \frac{Q}{2\pi r^2}}$$

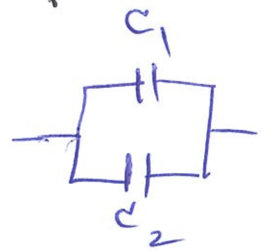
$$\Rightarrow D_{r_1} = \frac{Q\epsilon_1}{2\pi r^2(\epsilon_1+\epsilon_2)} \quad \Rightarrow E_{r_1} = \frac{Q}{2\pi r^2(\epsilon_1+\epsilon_2)}$$

$$V_{ab} = -\int_b^a \vec{E}\cdot d\vec{r} = -\frac{Q}{2\pi(\epsilon_1+\epsilon_2)}\int_b^a \frac{1}{r^2}\,dr = \frac{Q}{2\pi(\epsilon_1+\epsilon_2)}\left(\frac{b-a}{ab}\right)$$

$$C = \frac{Q}{V_{ab}} = 2\pi(\epsilon_1+\epsilon_2)\left[\frac{ab}{b-a}\right] = C_1 + C_2$$

Parallel capacitor

$$\Rightarrow C_1 = 2\pi\epsilon_1\left(\frac{ab}{b-a}\right), \quad C_2 = 2\pi\epsilon_2\left(\frac{ab}{b-a}\right) \Rightarrow$$



$C_{eq.} = C_1 + C_2$

Poisson's & Laplace's eqns: charge distribution ~~often~~ is not specified in the medium. We must find the e-field first, before we can find charge distributions.

NO ρ; $\Rightarrow E \to \rho$

– or we have problems that involve boundary surfaces on which either ρ_s or potential is specified. $\Rightarrow$ boundary value problems.

$\Rightarrow$ Alternative method: In linear medium:

$\vec{D} = \epsilon\vec{E}$, $\vec{\nabla}\cdot(\epsilon\vec{E}) = \rho_v$ → volume charge density

$\vec{E} = -\vec{\nabla}V \to \vec{\nabla}\cdot(-\epsilon\vec{\nabla}V) = \rho_v \Rightarrow \epsilon\vec{\nabla}\cdot(\vec{\nabla}V) + \vec{\nabla}V\cdot\vec{\nabla}\epsilon = -\rho_v$

$\Rightarrow \epsilon\nabla^2 V + \vec{\nabla}V\cdot\vec{\nabla}\epsilon = -\rho_v$ → 2nd order Partial D.E. position

For this eqn we have to know ~~b...~~ boundary conditions & the functional dependence of ρ_v & ϵ.

for homogeneous medium: $\epsilon = \text{const} \Rightarrow \vec{\nabla}\epsilon = 0$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{Poisson's eqn.}}$$

Potential distribution in a region depends upon the local charge distribution.

answer: $V = \dfrac{1}{4\pi\epsilon}\displaystyle\int_v \frac{\rho_v(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|}$

Some Problems in electrostatics involve charge distributions on the surface of conductors. $\Rightarrow \rho_v = 0 \Rightarrow \boxed{\nabla^2 V = 0 \quad \text{Laplace's eqn.}}$

$\Rightarrow$ For charge-free region we seek a Pot. function V that satisfies Laplace's eqn. subjected to B.C. $\Rightarrow$ with known $V \Rightarrow \vec{E} = -\vec{\nabla}V$

For a Linear, homogeneous, charge-free region: $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow$ class I field.

. solution to Laplace's eq is unique.

EX. Two metal Plates (A, d, ϵ). $V_{z=d} = V_0$, $V_{z=0} = 0$

a) Pot. distribution $V(z)$=?

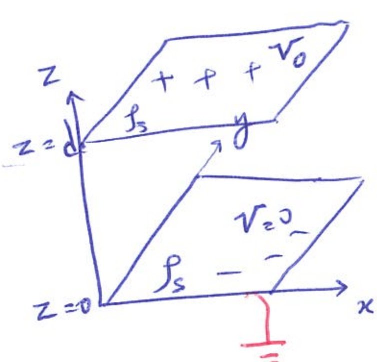
$V_x = V_y$ for all space.

conductors are equipotential in xy plane.

b) $\vec{E}$=?

c) ρ_s on plate?

d) C =?



$\Rightarrow$ charge-Free region: $\nabla^2 V = 0$ LaP.

$$\frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow \boxed{V = az + b} \quad a, b \text{ from B.C.}$$

$z=0,\ V=0 \Rightarrow b = 0$

$z=d,\ V=V_0 \Rightarrow a = \dfrac{V_0}{d}$ $\Rightarrow \boxed{V = \dfrac{z}{d}V_0}$ linear.

b) $\vec{E} = -\vec{\nabla}V = -\hat{z}\frac{\partial V}{\partial z} = -\frac{V_0}{d}\hat{z}$

$\vec{D} = \epsilon\vec{E} = -\frac{\epsilon V_0}{d}\hat{z}$

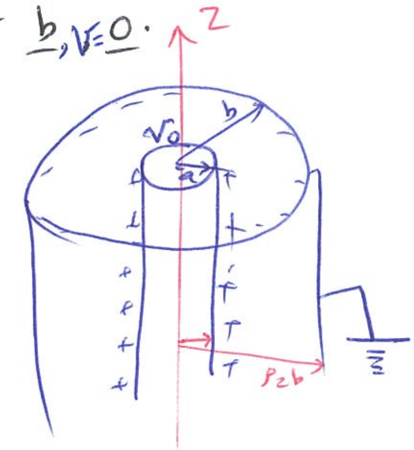
c) $\underset{z=0}{D_{n1}} = \rho_{s1} \Rightarrow -\frac{\epsilon V_0}{d} = \rho_{s1} \Rightarrow \boxed{\rho_{s1}\Big|_{z=0} = -\frac{\epsilon V_0}{d}}$ $(\vec{D}\cdot\hat{n}), (\hat{n}=\hat{z})$

$\underset{z=d}{D_{n2}} = \rho_{s2} \Rightarrow \boxed{\rho_{s2}\Big|_{z=d} = \frac{\epsilon V_0}{d}}$ $(\hat{n}=-\hat{z})$

$\begin{cases} Q = \rho_s A = \frac{\epsilon V_0 A}{d} & \text{uppler plen} \\ Q = -\frac{\epsilon V_0 A}{d} & \text{lower plane.} \end{cases}$

d) $C = \frac{Q}{V_0} = \frac{\epsilon A}{d}$.

Ex: coaxial cable: Inner conductor $\underline{a}, \underline{V_0}$, outer $\underline{b}, \underline{V=0}$.



a) Pot. distr. $\nabla^2 V = 0$; $V = V(\rho)$

b) ρ_s of inner. conductor: $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$

c) $C/L = ?$

$\Rightarrow V = g\ln\rho + d$

$\rho = b, V = 0 \Rightarrow d = -g\ln b \Rightarrow V = g\ln\left(\frac{\rho}{b}\right)$

$\rho = a, V = V_0 \rightarrow g = V_0/\ln(a/b) \Rightarrow$ for $a \le \rho \le b$: $V = V_0\frac{\ln(\rho/b)}{\ln(a/b)}$.

$\begin{cases} \vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial\rho}\hat{\rho} = \frac{V_0\hat{\rho}}{\rho\ln(b/a)}. \\ \vec{D} = \epsilon\vec{E} = \frac{\epsilon V_0}{\rho\ln(b/a)}\hat{\rho} \end{cases}$

$\vec{D}\cdot\vec{n} = \rho_s \Rightarrow \boxed{\rho_s = \frac{\epsilon V_0}{a\ln(b/a)}}$ $\vec{n} = \hat{\rho}$, $\rho = a$

$\boxed{Q/L = \frac{2\pi\epsilon V_0}{\ln(b/a)}} \Rightarrow \boxed{C/L = \frac{2\pi\epsilon}{\ln(b/a)}}$

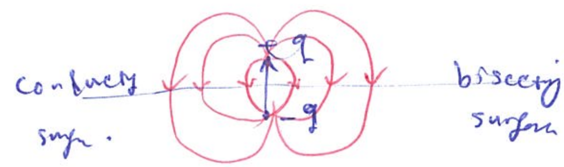
method of images:

The effect of surrounding on calculation of E-field.

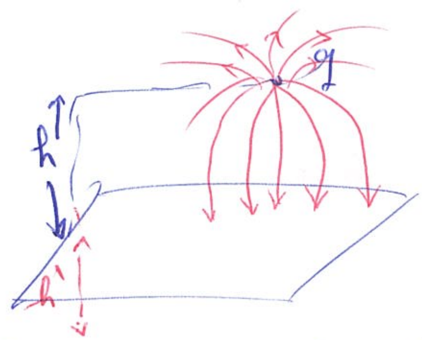
effect of → earth on the fields from an open-wire transmission line can not be ignored

→ field patterns of transmitting & receiving antennas are greatly modified by the presence of the conducting bodies on which they are mounted.

⇒ For conducting plane: Look at dipole:



if we ~~remove~~ put a conducting plane on bisecting surface → The field pattern of the dipole remains unchanged. → Now removing the $-q$ below the conducting plane, the field distribution in the region above the plane still remains the same. → Total charge induced on the surface of the conductor is $-q$.

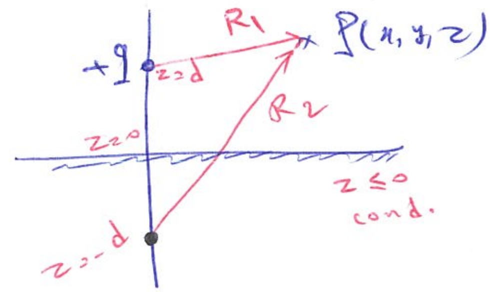


Conversely: If we are given a point charge q at h above a conducting plane of ∞ extent. ⇒ we can find V & E at any point above the plane by ignoring the plane & imaging a charge $-q$ at the same distance away on the other side of the plane. → the imaginary charge $-q$ is said to be the image of the real charge q.

⇒ For a curved surface $q_{im.} \neq q$, $h' \neq h$.

EX: A q is located above a conducting plane (∞) extent ~~depth~~.
$$\begin{cases} V = ? \\ Q_{ind} \overset{?}{=} -q \end{cases}$$

$$\begin{cases} q\,(0,0,d) \\ \text{im. charge: } -q\,(0,0,-d) \rightarrow \text{ignore the cond. plane.} \end{cases}$$



$$V = \frac{q}{4\pi\epsilon}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \quad \text{for } P(x,y,z),\ z \geq 0$$

$$R_1 = \left[x^2 + y^2 + (z-d)^2\right]^{1/2}$$

$$R_2 = \left[x^2 + y^2 + (z+d)^2\right]^{1/2}$$

on the surface of the conductor: $z = 0 \Rightarrow R_1 = R_2 \Rightarrow \boxed{V = 0}$

$$\vec{E} = -\vec{\nabla} V = -\frac{q}{4\pi\epsilon}\left[\left(\frac{x}{R_2^3} - \frac{x}{R_1^3}\right)\hat{x} + \left(\frac{y}{R_2^3} - \frac{y}{R_1^3}\right)\hat{y} + \left(\frac{z+d}{R_2^3} - \frac{z-d}{R_1^3}\right)\hat{z}\right]$$

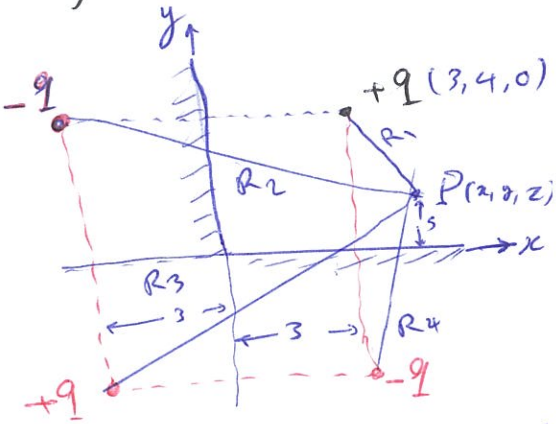
on the surface of cond $(R_1 = R_2) \Rightarrow \boxed{\vec{E} = -\frac{2qd}{4\pi\epsilon R^3}\hat{z}}$; $R = (x^2 + y^2 + d^2)^{1/2}$

$$\vec{D} = \epsilon\vec{E} = -\frac{2qd}{4\pi R^3} \Rightarrow \vec{D}\cdot\underbrace{\hat{n}}_{\hat{z}} = \rho_s \Rightarrow \boxed{\rho_s = -\frac{2qd}{4\pi R^3}}$$

$$Q_{ind} = \int_s \rho_s\, ds = -\frac{2qd}{4\pi}\int_0^\infty \frac{\rho\, d\rho}{(\rho^2 + d^2)^{3/2}}\int_0^{2\pi} d\phi = -q.$$

Ex: Two ∞ intersecting planes ($\theta = 90^\circ$) & a q of 100 nC at (3,4,0).

calculate $V(3,5,0)$ & $\vec{E}(3,5,0) = ?$



# of images for two parallel conducting planes = ∞

if angle θ is a submultiple of $360^\circ \Rightarrow$ # is finite.

$$n = \frac{360^\circ}{\theta} \Rightarrow \text{\# images} = n-1.$$

Here:

$$n = \frac{360}{90} = 4 \Rightarrow \text{\# images} = 3.$$

$$V(P) = \sum_{i=1}^{4} k\frac{q_i}{R_i} = \underbrace{kq}_{900}\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right];$$

$$\begin{cases} R_1 = [(x-3)^2+(y-4)^2+(z-0)^2]^{1/2} \\ R_2 = [(x+3)^2+(y-4)^2+(z-0)^2]^{1/2} \\ R_3 = [(x+3)^2+(y+4)^2+(z-0)^2]^{1/2} \\ R_4 = [(x-3)^2+(y+4)^2+(z-0)^2]^{1/2} \end{cases}$$

$$\Rightarrow \boxed{V(3,5,0) = 735.2\ V}$$

$$\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$$

$$\begin{cases} \frac{\partial V}{\partial x} = 900\left[-\frac{x-3}{R_1^3} + \frac{x+3}{R_2^3} - \frac{x+3}{R_3^3} + \frac{x-3}{R_4^3}\right] = 19.8 \quad \text{at } P(3,5,0). \\ \frac{\partial V}{\partial y} = -891.36 \\ \frac{\partial V}{\partial z} = 0 \end{cases}$$

$$\Rightarrow \boxed{\vec{E} = -19.8\hat{x} + 891.36\hat{y}\ \ V/m}$$