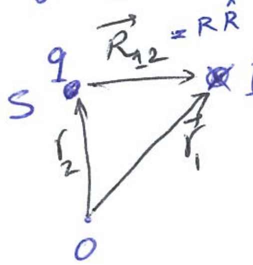
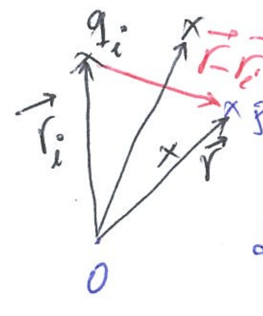
E-field at any Point P due to charge q at S :

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} = \frac{q}{4\pi\epsilon_0 R^2}\hat{R}$$



$$\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$



$\vec{r} - \vec{r}_i = \vec{R}_i$

distance vector

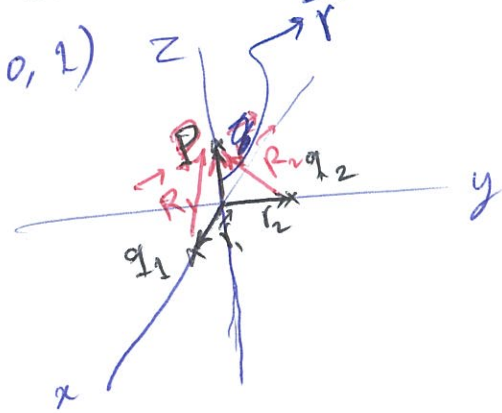
EX : Two Point charges $q_1 = 20\,nC$, $q_2 = -20\,nC$ are situated at $q_1(1, 0, 0)$ & $q_2(0, 1, 0)$ in free space. $\vec{E}(0, 0, 1) = ?$

$$\vec{R}_1 = \vec{r} - \vec{r}_1 \equiv (0, 0, 1) - (1, 0, 0) = (-1, 0, 1)$$

Point P

$$\Rightarrow \vec{R}_1 = -\hat{x} + \hat{z}$$

$$R_1 = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} = 1.414 \text{ m}$$



$$\vec{R}_2 = \vec{r} - \vec{r}_2 \equiv (0, 0, 1) - (0, 1, 0)$$

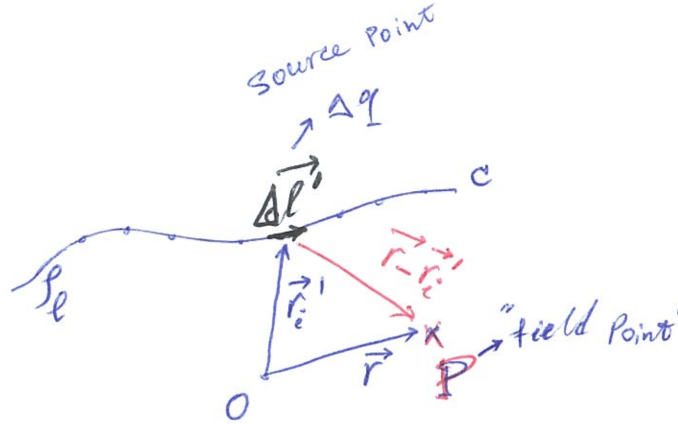
$$\Rightarrow \vec{R}_2 = (0, -1, 1) = -\hat{y} + \hat{z}$$

$$R_2 = \sqrt{(-1)^2 + 1^2} = \sqrt{2} = 1.414 \text{ m}$$

$$\vec{E}_P = \frac{q_1}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|^3} + \frac{q_2}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_2}{|\vec{r} - \vec{r}_2|^3} = \frac{1}{4\pi\epsilon_0}\left[q_1 \frac{-\hat{x} + \hat{z}}{1.414^3} + q_2 \frac{-\hat{y} + \hat{z}}{1.414^3} \right] \; ; q_2 = -q_1$$

$$= 9 \times 10^9 \left[\frac{q_1}{1.414^3} \left(-\hat{x} + \cancel{\hat{z}} + \hat{y} - \cancel{\hat{z}} \right) \right] = 63.67 \, (-\hat{x} + \hat{y}) \left[\text{V/m} \right]$$

Distributions of charges:



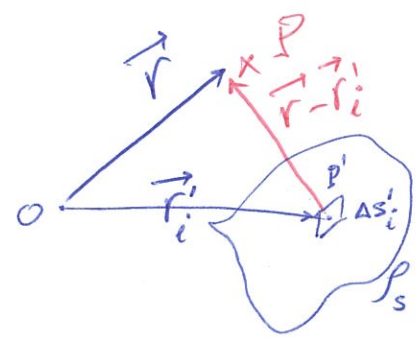
- Line charge density: = charge per unit length

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l}$$

$$\rightarrow \vec{E} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\Delta q_i}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}_i'}{|\vec{r}-\vec{r}_i'|^3} \quad \Leftarrow \Delta q_i = \rho_l \Delta l_i'$$

$$= \frac{1}{4\pi\epsilon_0}\int_c \frac{\rho_l(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, dl'$$

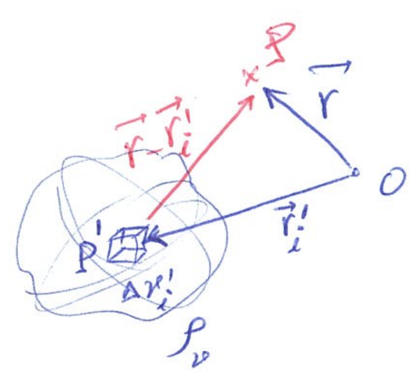
$$\begin{cases} \text{Prime} \rightarrow \text{Points on charge distribution} \\ \text{unprime} \rightarrow \text{Filed Point } \underline{\text{P}} \end{cases}$$



- surface charge density:

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_s \rho_s \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, ds'$$



- volume charge density:

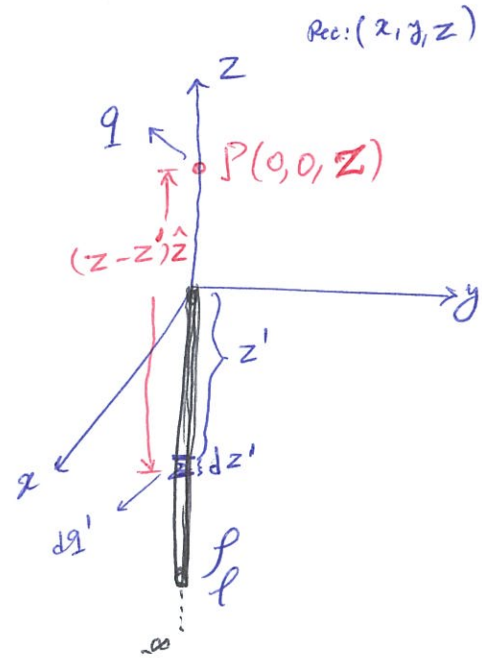
$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, dv'$$

EX1: Linear charge distribution, Rectangular coordinate system:

A semi-infinite line from $-\infty$ to 0 along z-axis carries a uniform charge $100 \ nC/m$. find $E(0,0,2)$?

Rec: (x, y, z)



$$dq' = \rho_\ell \, dz'$$

$$\vec{r} - \vec{r}' = (z\hat{z} - z'\hat{z}) = (z - z')\hat{z}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{0} \frac{\rho_\ell (z-z')\hat{z}}{|z-z'|^3} \, dz'$$

$$= \hat{z} \frac{\rho_\ell}{4\pi\epsilon_0} \int_{-\infty}^{0} \frac{dz'}{(z-z')^2} = \hat{z} \frac{\rho_\ell}{4\pi\epsilon_0} \left(\frac{1}{z-z'} \right)_{-\infty}^{0}$$

$$= \hat{z} \frac{\rho_\ell}{4\pi\epsilon_0} \left[\frac{1}{z-0} - \xrightarrow{0} \frac{1}{z+\infty} \right] = \hat{z} \frac{\rho_\ell}{4\pi\epsilon_0 \, z}$$

$; \ \frac{1}{4\pi\epsilon_0} = 9 \times 10^9$

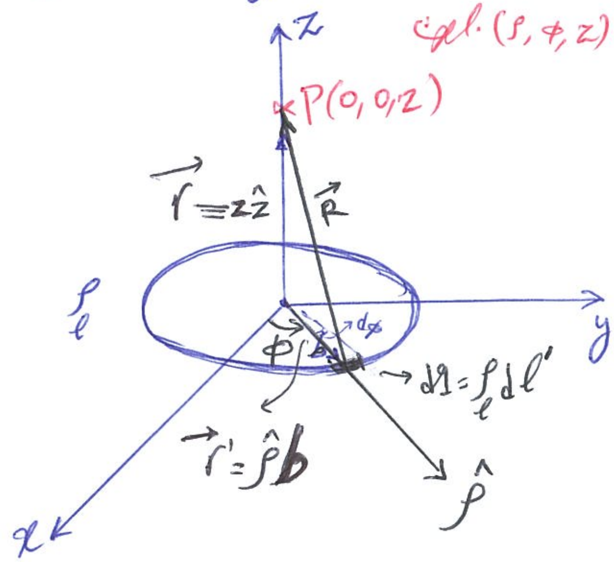
$$\vec{E}_P = 9 \times 10^{-9} \ \frac{100 \times 10^{-9}}{2} \hat{z} = 450 \, \hat{z} \ [V/m].$$

If $q = 1 \, \mu C$, what is the force acting on q:

$$\vec{F} = q\vec{E} = (1 \times 10^{-6})(450\,\hat{z}) = 450\,\hat{z} \ \mu N.$$

EX2: $\overline{\text{2-1}}$ Linear charge density, cylindrical coordinate system:

A ring of uniformly distributed charge with radius (b). Find the field intensity at any point on the axis of the ring:



$$dq = \rho_\ell \, dl' = \rho_\ell (b\, d\phi')$$

$$\vec{r} - \vec{r}' \equiv \vec{R} = z\hat{z} - b\hat{\rho} \quad ; \quad |\vec{R}| = \sqrt{b^2 + z^2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\phi'=0}^{2\pi} \frac{\rho_\ell \vec{R}}{|\vec{R}|^3} \, b\, d\phi'$$

$$= \frac{\rho_\ell}{4\pi\epsilon_0} \int_0^{2\pi} \frac{b\, d\phi'}{[b^2+z^2]^{3/2}} (-b\hat{\rho} + z\hat{z})$$

$$= \frac{\rho_\ell b}{4\pi\epsilon_0} \frac{1}{[b^2+z^2]^{3/2}} \left[-b\int_0^{2\pi} d\phi' \, \hat{\rho} + z\int_0^{2\pi} d\phi' \, \hat{z} \right] \rightarrow \int_0^{2\pi} d\phi' = 2\pi$$

(Symmetry)

$$\hat{\rho} = \hat{x}\cos\phi' + \hat{y}\sin\phi'$$

$$\int_0^{2\pi} \hat{\rho}\, d\phi' = \hat{x}\int_0^{2\pi} \cos\phi'\, d\phi' + \hat{y}\int_0^{2\pi} \sin\phi'\, d\phi' = 0$$

$$\vec{E} = \frac{\rho_\ell b z}{2\epsilon_0 [b^2+z^2]^{3/2}} \hat{z}$$

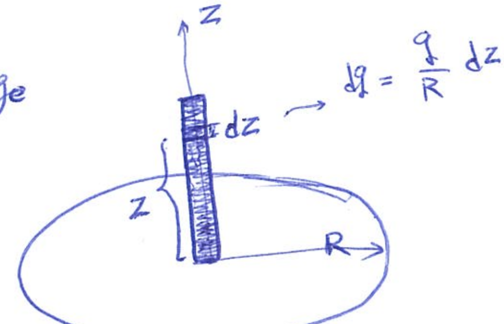
$\Rightarrow$ if $z=0 \Rightarrow \vec{E} = 0 \Rightarrow$ why?

↓

symmetry

34-1 Example

A thin rod of length R and ~~a~~ with uniform charge q is located on the axis of a thin disc of radius R and uniform charge Q. One end of rod is connected to the center of disc. Calculate the force exerted to the rod due to the disc.



we calculate the force exerted on a point charge on the rod, then move this point along the rod to calculate the total force.

$$E_Z = \frac{\rho_s}{2\epsilon_0}\left(1 - \frac{Z}{\sqrt{R^2+Z^2}}\right)$$

field of disc on Z.

$$= \frac{Q}{2\pi\epsilon_0 R^2}\left(1 - \frac{Z}{\sqrt{R^2+Z^2}}\right)$$

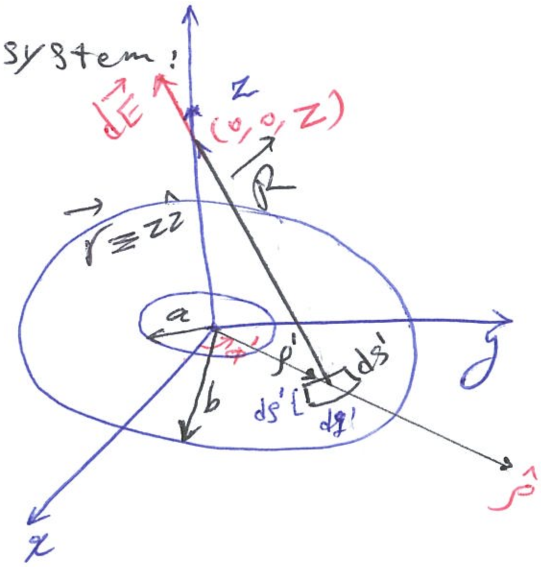
$$dF = E_Z\, dq = E_Z \frac{q}{R} dz = \frac{Q}{2\pi R^2 \epsilon_0}\left(1 - \frac{Z}{\sqrt{R^2+Z^2}}\right)\frac{q}{R} dz$$

$$F = \int_{z=0}^{R} dF = \frac{Qq}{2\pi R^3 \epsilon_0}\int_0^R \left(1 - \frac{Z}{\sqrt{R^2+Z^2}}\right) dz$$

$$= \frac{Qq}{2\pi R^3 \epsilon_0}\left[R - \int_0^R \frac{Z}{\sqrt{R^2+Z^2}} dz\right]$$

$$= \frac{Qq}{2\pi R^3 \epsilon_0}\left[R - \sqrt{Z^2+R^2}\Big|_0^R\right] = \frac{Qq}{2\pi\epsilon_0 R^3}(2-\sqrt{2})^N$$

EX3: surface charge density, cylindrical system:



A thin annular disc of inner radius $\underline{a}$, outer $\underline{b}$ with ρ_s. Determine $\vec{E}$ intensity at any point on z-axis ($z \geq 0$):

$$dq' = \underbrace{\rho_s (d\rho')(\rho' d\phi')}_{=\rho_s ds'} \Longleftarrow ds' = \rho' d\rho' d\phi'$$

$$\vec{R} = -\rho'\hat{\rho} + z\hat{z} \quad , |\vec{R}| = \left[\rho'^2 + z^2\right]^{1/2}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\rho'=a}^{b} \int_{\phi'=0}^{2\pi} \frac{\rho_s \vec{R}}{|\vec{R}|^3} \rho' d\rho' d\phi'$$

$$= \frac{\rho_s}{4\pi\epsilon_0} \int_{\rho'=a}^{b} \int_{\phi'=0}^{2\pi} \frac{\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}} \left[-\rho'\hat{\rho} + z\hat{z}\right]$$

with the same symmetry reason there is no E-filed in $\hat{\rho}$ direction

$$\int_0^{2\pi} \hat{\rho}\, d\phi' = 0$$

$$\Rightarrow \vec{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_{\rho'=a}^{b} \int_{\phi'=0}^{2\pi} \frac{\rho' d\rho' d\phi'}{\left[\rho'^2 + z^2\right]^{3/2}} z\hat{z}$$

$$= \frac{\rho_s z}{4\pi\epsilon_0} \underbrace{\int_{\rho'=a}^{b} \frac{\rho' d\rho'}{\left[\rho'^2 + z^2\right]^{3/2}}}_{\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{(b^2+z^2)^{1/2}}} \underbrace{\int_{\phi'=0}^{2\pi} d\phi'}_{2\pi} \hat{z}$$

$$\Rightarrow \vec{E} = \frac{\rho_s z}{2\epsilon_0} \left[\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{(b^2+z^2)^{1/2}}\right] \cdot \hat{z}$$

$$\left\{\begin{array}{l} b \to \infty \\ a \to 0 \end{array}\right. \Rightarrow \boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0} \hat{z}}$$

Before going to spherical system →

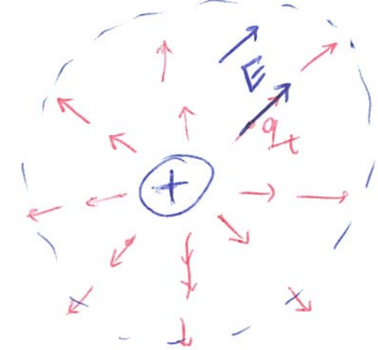
Electric Flux :

Line of force or Flux line: Place a test charge at one Point in $\vec{E}$ & allow it to move.

$\Rightarrow$ # of line of force = q in coulombs. → Field lines represent the E-flux.

↳ magnitude.

→ $\vec{E}$-field intensity is tangential to the lines of E-flux →
- indep. of medium
- $|E| \propto q$
- uniformly distributed
- e-flux density (flux/A) $\propto \frac{1}{R^2}$



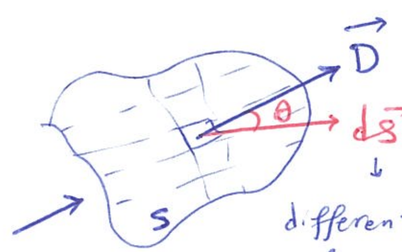
E-flux density: $\vec{D} = \epsilon_0 \vec{E}$

↳ Permittivity of free space.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{r} \Rightarrow \vec{D} = \frac{q}{4\pi r^2} \hat{r} \left[\frac{C}{m^2}\right]$$

E-flux: $\Psi = \int_s \vec{D} \cdot d\vec{S}$

↳ if $\vec{D}$ & $d\vec{s}$ are in the same direction → Ψ is max.

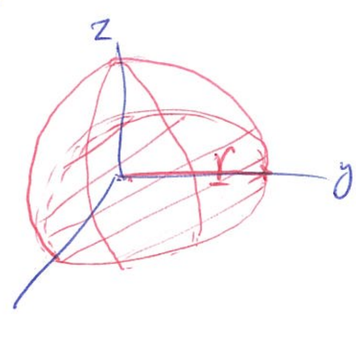


↓ differential surface element on surface S.

EX 3.7 : $\vec{D} = 10\hat{r} + 5\hat{\theta} + 3\hat{\phi} \left[\frac{mC}{m^2}\right]$, what is Ψ passing through the surface bounded by $z \geq 0$ and $x^2+y^2+z^2 = 36 = r^2 \Rightarrow r = 6$ m

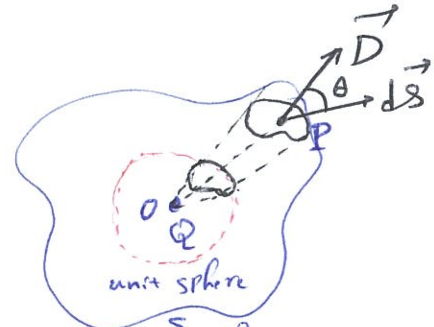
$d\vec{s} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$ → $\vec{D} \cdot d\vec{S} = (10)(36) \sin\theta \, d\theta \, d\phi$

$$\Psi = \int_s \vec{D} \cdot d\vec{S} = 360 \int_{\theta=0}^{\pi/2} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi = 720\pi \ [mC].$$



Gauss's law: the net outward flux passing through a closed surface is equal to the total charge enclosed by that surface:

$$\oint_S \vec{D}\cdot d\vec{S} = Q$$



arbitrary surface.

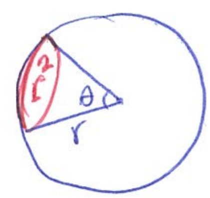
prove: A point charge inside S:

$$\vec{D} = \frac{Q}{4\pi R^2}\hat{R} \quad , \vec{R} = \vec{r} - \vec{r}\,' = R\hat{R}$$

$dl = R\,d\phi$

$$\Psi = \oint_S \vec{D}\cdot d\vec{S} = \frac{Q}{4\pi}\oint_S \frac{\hat{R}\cdot\hat{N}\,dS}{R^2} \rightarrow R^2\,d\Omega$$

$d\Omega$ ↓ solid angle (3D-angle)



$$= \frac{Q}{4\pi}\oint_S d\Omega = Q \checkmark$$

→ The total e-flux emanating from a closed surface is numerically equal to the net $+Q$ inside the closed surface.

integral form of Gauss's law

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\epsilon_0} \quad ; \quad \oint_S \vec{D}\cdot d\vec{S} = \int_v \rho_v\,dv . \quad \text{volume charge distribution}$$

$$= \int_S \rho_s\,dS . \quad \text{surface} — —$$

$$= \int_l \rho_l\,dl . \quad \text{linear} — —$$

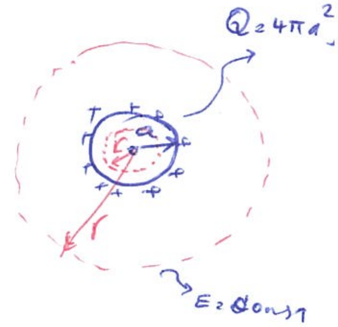
→ If we know $\vec{E}$ or $\vec{D}$ on all points of a surface → we can find the total charge enclosed in that surface.

→ If the charge distribution is symmetric, and a convenient surface can be chos[en] on which the e-flux density is constant ⟹ Gauss's law make it simple.

Using Divergence theorem: $\oint_S \vec{D}\cdot d\vec{S} = \int_v \vec{\nabla}\cdot\vec{D}\,dv = Q = \int_v \rho_v\,dv . \rightarrow \boxed{\vec{\nabla}\cdot\vec{D} = \rho_v}$ Diff. form of Gauss

$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_v}$ ⇒ Lines of E-flux emanate from any point in space at which there exists a (+) charge density. If ~~there~~ there exist $-Q$, the lines of e-flux converge toward that point.

Ex 3.9: A charge is uniformly distributed over a spherical surface. ($R=a$). find $\vec{E}$ everywhere in space.

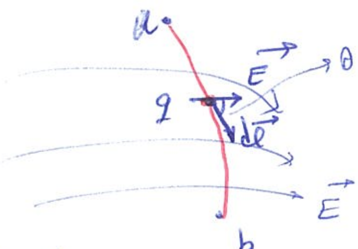


$r<a$: $\oint \vec{E}\cdot d\vec{s} = \frac{Q_{enc.}}{\epsilon_0} \Rightarrow E(4\pi r^2) = 0 \Rightarrow E = 0$

$r \geq a$: $E(4\pi r^2) = \frac{\rho_s 4\pi a^2}{\epsilon_0} \Rightarrow \vec{E} = \frac{\rho_s a^2}{\epsilon_0 r^2}\hat{r}$

Electric Potential: A scalar field to simplify the calculation.

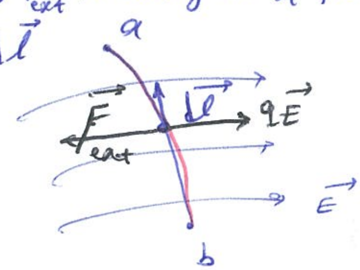
$+q_t$ in an E-field $\vec{E}$:



an external force just balance the E-force:

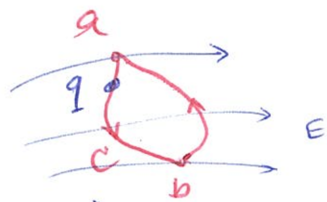
Total work done by F_{ext} in moving the q_t from b to a:

$W_{ab} = -q\int_b^a \vec{E}\cdot d\vec{l}$



$dw_e = \vec{F}\cdot d\vec{l} = q\vec{E}\cdot d\vec{l}$

$w_e = q\int_a^b \vec{E}\cdot d\vec{l}$



→ If we move the charge around a closed path → $W_{tot} = 0$ ⇒ $\boxed{\oint_c \vec{E}\cdot d\vec{l} = 0}$

E-field under static conditions is irrotational or conservative.

$W_{ab} = -q\int_b^a \vec{E}\cdot d\vec{l} = q\int_b^a \underbrace{\vec{\nabla}V\cdot d\vec{l}}_{dV}$ ⇐ $\boxed{\vec{E} = -\vec{\nabla}V}$ ⇐ $\boxed{\vec{\nabla}\times\vec{E} = 0}$ by definition.

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$\to W_{ab} = -q \int_b^a \vec{E}\cdot d\vec{l} = q \int_{V_b}^{V_a} dV = q\,[V_a - V_b] = q\,V_{ab}$ → Potential of $\underline{a}$ withrespect $\underline{b}$.

V_a: e-Potential at Poin a.

V_{ab}: Potential difference between a & b.

$\to$ if $W_{ab} > 0 \Rightarrow V_a > V_b$. → external force is pushing $+q$ against $\vec{E}$.

⇒ the potential energy of the charge is increasing.

$\to \vec{E} = -\vec{\nabla} V \Rightarrow$ The work done in moving a $\underline{+q}$ against $\vec{E}$ is equal to the increase in the Potential energy of the charge.

$-$ sign

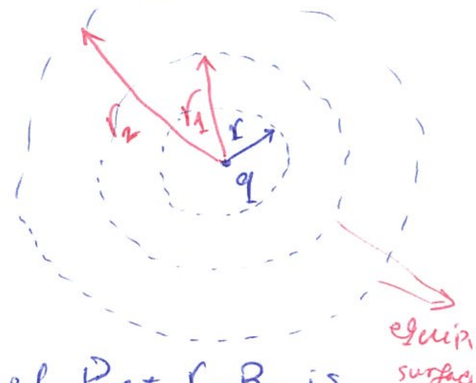
Potential difference : change in Pot energy per unit charge in the limit $q \to 0$:

$$V_{ab} = \lim_{q\to 0} \frac{W_{ab}}{q} = -\int_b^a \vec{E}\cdot d\vec{l} \qquad [J/C] \equiv [\text{Volt}]$$

$$[\vec{E}] \equiv [V/m].$$

EX : Find ΔV (Pot. difference) between two points due to a point charge q:

$d\vec{l} = d\vec{r}$



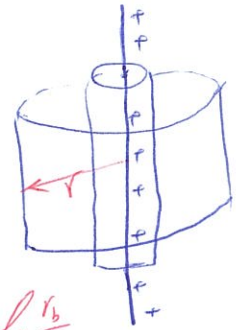
$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \quad \to V_{ab} = -\int_{r_2}^{r_1} \frac{q}{4\pi\epsilon_0 r^2}\,dr$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

if $r_2 \to \infty$ → reference point at infinity ⇒ Potential of P at $r_1 = R$ is called "Absolute Potential". ⇒ $V_a = \dfrac{q}{4\pi\epsilon_0 R}$ → $\underline{V_a = \text{const for } R = \text{const}}$

Point charge $\boxed{V_q = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r}}$ $\to dV = \dfrac{1}{4\pi\epsilon_0}\dfrac{dq}{r} \Rightarrow V = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \dfrac{dq}{r}$

Ex: For cylindrical surfaces: Equipot. surfaces for a uniformly charged line:



$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi r \ell) = \frac{\ell \rho_\ell}{\epsilon_0} \Rightarrow \vec{E} = \frac{\rho_\ell}{2\pi\epsilon_0 r}\hat{r} \quad ; d\vec{\ell} = dr\,\hat{a}_r$$

$$V_{ab} = -\int_{r_b}^{r_a} \vec{E}\cdot d\vec{\ell} = -\int_{r_b}^{r_a} \frac{\rho_\ell}{2\pi\epsilon_0 r}\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln r \Big|_{r_b}^{r_a} = \frac{-\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_a}{r_b} = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_b}{r_a}$$

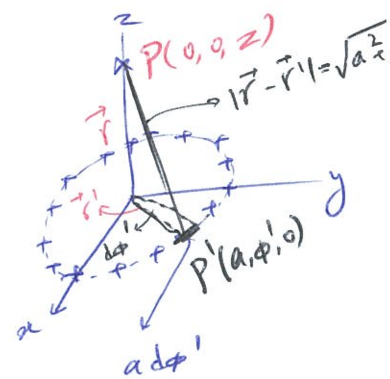
ref. at $r_a = R \Rightarrow V_R = 0 \rightarrow$ Grounding!

$$\Rightarrow V = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_v(\vec{r}\,')\,dv'}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_s \frac{\rho_s(\vec{r}\,')\,ds'}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_c \frac{\rho_\ell(\vec{r}\,')\,d\ell'}{|\vec{r}-\vec{r}\,'|}$$

Ex: A charged ring of radius a; Find V & E at any point on the axis of ring.



$$V(z) = \frac{1}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi} \frac{\rho_\ell\, a\, d\phi'}{(a^2+z^2)^{1/2}} = \frac{\rho_\ell\, a}{2\epsilon_0\sqrt{a^2+z^2}}$$

$$V(z=0) = \frac{\rho_\ell}{2\epsilon_0}$$

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V(z)}{\partial z}\hat{z} = \frac{\rho_\ell\, a}{2\epsilon_0}\left[\frac{z}{(a^2+z^2)^{3/2}}\right]\hat{z}$$

$\Rightarrow E(z=0) = 0 \rightarrow$ symmetry.