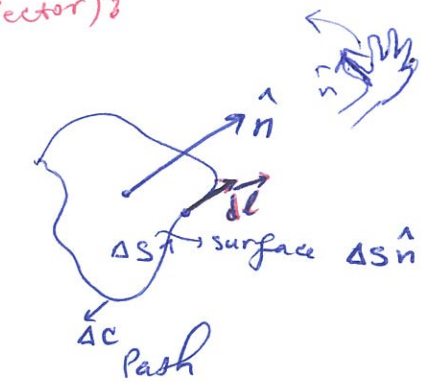
Curl of a vector field:

circulation of $\vec{F}$: integral of a vector field $\vec{F}$ around a closed Path.

curl of $\vec{F}$: measure of the circulation of $\vec{F}$. (a vector);



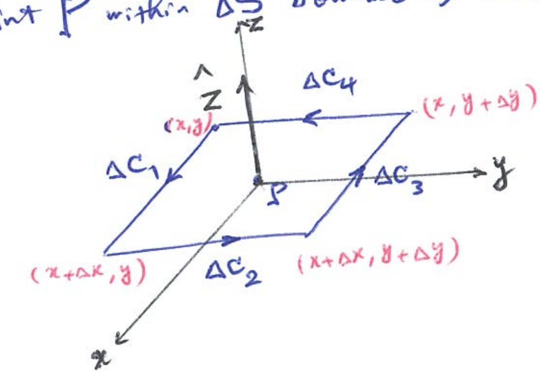
definition: component of the curl of $\vec{F}$ parallel to the surface normal $\hat{n}$, in the limit $\Delta S \to 0$:

$$(\text{curl}\,\vec{F})\cdot\hat{n} = \lim_{\Delta S\to 0} \frac{1}{\Delta S}\oint_{\Delta C} \vec{F}\cdot d\vec{l}$$

→ we can find three components of curl$\vec{F}$ in any c.s.

rectangular c.s.: $\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$ at point P within ΔS bounded by ΔC.

To calculate the z-comp. of curl $\vec{F}$:



$$\oint_{\Delta C}\vec{F}\cdot d\vec{l} = \int_{\Delta C_1}\vec{F}\cdot d\vec{l} + \int_{\Delta C_2}\vec{F}\cdot d\vec{l} + \int_{\Delta C_3}\vec{F}\cdot d\vec{l} + \int_{\Delta C_4}\vec{F}\cdot d\vec{l}$$

$$\int_{\Delta C_1}\vec{F}\cdot d\vec{l} = \int_x^{x+\Delta x}(F_x\hat{x} + F_y\hat{y} + F_z\hat{z})\cdot(dx\,\hat{x}) = (F_x\,\Delta x)\Big|_y$$

- based on mean value theorem
  we assume that $F_x \simeq$ const. from $x \to x+\Delta x$.
- calculate F_x at y.

$$\int_{\Delta C_2}\vec{F}\cdot d\vec{l} = \int_y^{y+\Delta y}(F_x\hat{x} + F_y\hat{y} + F_z\hat{z})\cdot(dy\,\hat{y}) = F_y\,\Delta y\Big|_{x+\Delta x}$$

$$\int_{\Delta C_3}\vec{F}\cdot d\vec{l} = \int_{x+\Delta x}^{x}(F_x\hat{x} + F_y\hat{y} + F_z\hat{z})\cdot(dx\,\hat{x}) = -(F_x\,\Delta x)\Big|_{y+\Delta y}$$

$$\int_{\Delta C_4}\vec{F}\cdot d\vec{l} = \int_{y+\Delta y}^{y}\ldots\cdot(dy\,\hat{y}) = -(F_y\,\Delta y)\Big|_x$$

$$\to \oint_{\Delta C}\vec{F}\cdot d\vec{l} = (F_x\,\Delta x)\Big|_y - (F_x\,\Delta x)\Big|_{y+\Delta y} + (F_y\,\Delta y)\Big|_{x+\Delta x} - (F_y\,\Delta y)\Big|_x$$

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$$\begin{cases} \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (F_x \Delta x)\Big|_{y+\Delta y} - (F_x \Delta x)\Big|_{y} = \dfrac{\partial F_x}{\partial y} \Delta y \Delta x \\ \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} (F_y \Delta y)\Big|_{x+\Delta x} - (F_y \Delta y)\Big|_{x} = \dfrac{\partial F_y}{\partial x} \Delta x \Delta y \end{cases}$$

→ using taylor series expansion & neglecting higher order terms.

$$\Rightarrow \oint_{\Delta C} \vec{F} \cdot d\vec{l} = \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \underbrace{\Delta x \Delta y}_{\Delta S}$$

dividing $\Delta S = \Delta x \, \Delta y$

$$\Rightarrow \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \vec{F} \cdot d\vec{l} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

$\hat{n} \equiv \hat{z}$

$$\Rightarrow \boxed{(\text{curl} \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}} \Rightarrow$$

$$(\text{curl} \vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

$$(\text{curl} \vec{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

$$\Rightarrow \text{curl} \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$ in rectangular c.s.

$$= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \right)$$

$$= \vec{\nabla} \times \vec{F}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$ → we always use this form. (determinant form).

$$\vec{\nabla} \times \vec{F} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \partial/\partial \phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$ in cylindrical c.s.

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$ in spherical coordinate system.

physical significance of the curl of a vector field: it represents the circulation per unit area of the vector field around a small area of any shape.

→ its direction: normal to the plane of the surface.

→ rotational vector field: If the line integral of a vector field about a closed elementary path $\neq 0$ ⇒ the curl of the vector field is also nonzero.

→ like flow of water out of a tub or sink! (rotational velocity)

→ Irrotational or Conservative vector field: if Curl = 0

~~like force F~~ ~~$dw = \vec{F}\cdot d\vec{l}$~~ ~~work done by a force~~

if $\vec{\nabla}\times\vec{F} = 0 \Rightarrow \oint_C \vec{F}\cdot d\vec{l} = 0$

EX: $\vec{\nabla}\times(\vec{\nabla}f) = 0$ if $f = f(x,y,z)$.

→ $\vec{\nabla}f$ is an irrotational or conservative field.

or: If the curl of vector field is Zero, the vector field is the gradiant of a scalar function.

if $\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F} = \pm\vec{\nabla}f$; $\pm$ → depends on physical interpretation of f.

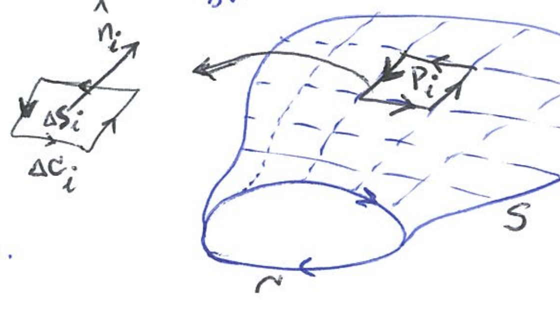
Stokes' theorem:

$$(\mathrm{curl}\,\vec{F})\cdot\hat{n} = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_{\Delta C}\vec{F}\cdot d\vec{l}$$

⇒ → we can find a relation between a finite but open surface area S bounded by a closed contour C.

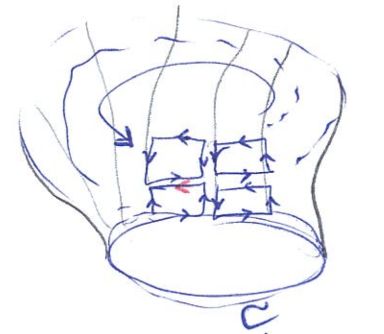
$$\int_{\Delta S_i}(\vec{\nabla}\times\vec{F})\cdot d\vec{S}_i = \oint_{\Delta C_i}\vec{F}\cdot d\vec{l} + \epsilon_i\,\Delta S_i$$

when $n\to\infty$, $\epsilon_i \to 0$.



$d\vec{S}_i = dS_i\,\hat{n}$

for the entire area:



only $\underline{C}$ remains.

$$\sum_{i=1}^{n} \int_{\Delta S_i} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}_i = \sum_{i=1}^{n} \oint_{\Delta C_i} \vec{F}\cdot d\vec{l} + \sum_{i=1}^{n} \epsilon_i \, \Delta S_i$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\Delta S_i} (\vec{\nabla}\times\vec{F})\cdot d\vec{S}_i = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$S \rightarrow$ open surface bounded by contour $\underline{C}$.

$\Rightarrow$ line integrals $\oint_{\Delta C_i} \vec{F}\cdot d\vec{l}$ along adjacent elementary areas cancel because the length vectors are directed in opposite directions. $\Rightarrow$ only integral over the path $\underline{C}$ will remain.

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \oint_{\Delta C_i} \vec{F}\cdot d\vec{l} = \oint_C \vec{F}\cdot d\vec{l}$$

$$\Rightarrow \boxed{\int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{l}}$$

S: open surface; C: closed line

* The normal component of the curl of a vector field over an area is equal to the line integral of the vector field along the curve bounding the area.

The Laplacian operator:

$\vec{\nabla}\times$, $\vec{\nabla}\cdot$, $\vec{\nabla}$: 1^{st} order differential operators.

∇^2 : 2^{nd} order diff. operator : Laplacian operator. = divergence of a gradient of a scalar function.

R.C.S.

$$\nabla^2 f = \vec{\nabla}\cdot(\vec{\nabla} f) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

c.c.s.

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

s.c.s.

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

Laplac's eqn: $\nabla^2 f = 0$;

$\nabla^2 \vec{F} = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{F})$ $\longrightarrow$ we use in field theory.

in R.C.S. : $\nabla^2 \vec{F} = \hat{x}\,\nabla^2 F_x + \hat{y}\,\nabla^2 F_y + \hat{z}\,\nabla^2 F_z$ $\longrightarrow$ $\nabla^2 \vec{F} = 0$ if & only if Laplacian of each component is independently Zero

$$\boxed{\text{Laplacian operator: } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Ex: $f = \frac{1}{r}$; $\nabla^2 f = ?$ $\longrightarrow$

we work in spherical c.s : $\nabla^2 f = \nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{r^2}\right)\right] = 0$

$\frac{1}{r}$ is a solution to Laplac's eqn. $\Leftarrow$

Green's theorem: $\underline{A}$ is single-valued, continuously differentiable vector every where in volume $\underline{v}$ & on its surface $\underline{S}$:

divergence theorem:

$$\int_v \vec{\nabla}\cdot\vec{A}\,dv = \oint_s \vec{A}\cdot d\vec{S} \quad ; \quad \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(\phi\,\vec{\nabla}\psi) = \vec{\nabla}\phi\cdot\vec{\nabla}\psi + \phi\,\nabla^2\psi$$

$$\boxed{\int_v \phi\,\nabla^2\psi\,dv + \int_v \vec{\nabla}\phi\cdot\vec{\nabla}\psi\,dv = \oint_s \phi\,\vec{\nabla}\psi\cdot d\vec{S}} \quad (1)$$

1st Green's identity

by interchanging ϕ & ψ : $\int_v \psi\,\nabla^2\phi\,dv + \int_v \vec{\nabla}\psi\cdot\vec{\nabla}\phi\,dv = \oint_s \psi\,\vec{\nabla}\phi\cdot d\vec{S}$ (2)

subtracting (1)-(2) $\Rightarrow$

$$\boxed{\int_v \left[\phi\,\nabla^2\psi - \psi\,\nabla^2\phi\right] dv = \oint_s \left[\phi\,\vec{\nabla}\psi - \psi\,\vec{\nabla}\phi\right]\cdot d\vec{S}}$$

2nd Green identity

if $\phi = \psi \Rightarrow \int_v \phi\,\nabla^2\phi\,dv + \int_v |\nabla\phi|^2\,dv = \oint_s \phi\,\vec{\nabla}\phi\cdot d\vec{S}$

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The uniqueness theorem: A vector field $\vec{A}$ is uniquely determined in a region if:

a) its divergence is specified throughout the region.

b) " curl — — —

c) " normal component is specified on the closed surface bounding the region.

assume there are 2 vectors satisfying above conditions ($\vec{A}$ & $\vec{B}$):

$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{B}$ & $\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{B}$, $\vec{A}\cdot d\vec{S} = \vec{B}\cdot d\vec{S}$ → on any differential surface $d\vec{S}$.

for every point in v.

$\vec{C} = \vec{A} - \vec{B} \Rightarrow \underbrace{\vec{\nabla}\cdot\vec{C} = 0}_{= \nabla\cdot A - \nabla\cdot B}$; $\vec{\nabla}\times\vec{C} = \vec{\nabla}\times\vec{A} - \vec{\nabla}\times\vec{B} = 0$; $\vec{C}\cdot d\vec{S} = 0$

we can write $\vec{C}$ as:

$\vec{C} = -\vec{\nabla} f$

$\vec{\nabla}\cdot(\vec{\nabla} f) = 0$ or $\nabla^2 f = 0$

$\vec{\nabla} f \cdot d\vec{S} = 0 \xleftarrow{C = -\vec{\nabla} f}$

$\Rightarrow \int_v \phi \nabla^2\phi \, dv + \int_v |\nabla\phi|^2 \, dv = \oint_S \vec{\nabla}\phi \cdot d\vec{S} \xRightarrow{\phi = f} \int_v |\nabla f|^2 \, dv = 0$

> 0

$\text{unique.} \Leftarrow \boxed{\vec{A} = \vec{B}} \Leftarrow \vec{C} = 0 \Leftarrow \nabla f = 0$

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classification of fields:

class I: $\vec{F}$ is a class I field everywhere in a region if: $\vec{\nabla}\cdot\vec{F} = 0$ & $\vec{\nabla}\times\vec{F} = 0$

$\vec{F} = -\vec{\nabla} f$

$\vec{\nabla}\cdot(-\vec{\nabla} f) = \nabla^2 f = 0$

To obtain fields of class I ($\vec{F}$), we need to solve Laplace's eqn with the conditions at the boundary of the region.

when we know $f \Rightarrow \vec{F} = -\vec{\nabla} f$. ⇒ Like Electrostatic fields in charge-free medium & Magnetic fields in current-free medium.

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class II : $\vec{F}$ is a class II field in a region if: $\vec{\nabla}\cdot\vec{F} \neq 0$ & $\vec{\nabla}\times\vec{F} = 0$

$\vec{F} = -\vec{\nabla} f$

Poisson's eqn.

$$\boxed{\nabla^2 f = -\rho} \Leftarrow \vec{\nabla}\cdot\vec{F} = \rho$$

ρ → a const or a known function within the region.

class II fields can be found by solving Poisson's eqn within the constraints of the boundary conditions. → like electrostatic field in a charged region.

class III : $\vec{\nabla}\cdot\vec{F} = 0$ & $\vec{\nabla}\times\vec{F} \neq 0$

$\vec{F} = \vec{\nabla}\times\vec{A}$

$\vec{\nabla}\times\vec{F} = \vec{J}$ → known vector field.

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{J}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \vec{J}$$

according to uniqueness theorem, for $\vec{A}$ to be a unique vector field, we must define its divergence.

Coulomb's gauge ← $\vec{\nabla}\cdot\vec{A} = 0$ ← arbitrary constraint

$$\Rightarrow \boxed{\nabla^2\vec{A} = -\vec{J}}$$ Poisson's vector eqn.

Like magnetic field within a current-carrying conductor.

class IV : $\vec{\nabla}\cdot\vec{F} \neq 0$, $\vec{\nabla}\times\vec{F} \neq 0$, but we can decompose $\vec{F}$ into 2 vector fields $\vec{G}$ & $\vec{H}$ such that $\vec{G}$ satisfy class III & $\vec{H}$ satisfies class II requirements:

$\vec{F} = \vec{G} + \vec{H}$, $\vec{\nabla}\cdot\vec{G} = 0$, $\vec{\nabla}\times\vec{G} \neq 0$, $\vec{\nabla}\times\vec{H} = 0$, $\vec{\nabla}\cdot\vec{H} \neq 0$

$\vec{G} = \vec{\nabla}\times\vec{A}$; $\vec{H} = -\vec{\nabla} f$

$$\Rightarrow \boxed{\vec{F} = \vec{\nabla}\times\vec{A} - \vec{\nabla} f}$$

Like: Hydrodynamic fields in a compressible medium.

Assignment #1 : <u>Exercises</u>:

2.2, 2.10, 2.12, 2.20, 2.28, 2.29, 2.41, 2.47

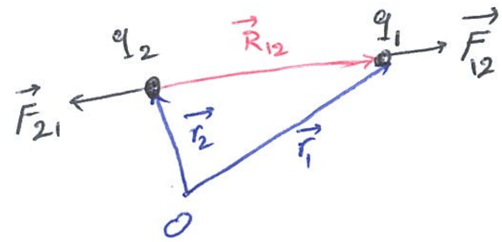
<u>problems</u>:

2.6, 2.15, 2.20, 2.23, 2.25, 2.27, 2.31, 2.36, 2.38, 2.44, 2.49

Electrostatics

Static Electric Fields: due to charges at rest.

Coulomb's law: electric force between two charged particles:



$$\vec{F}_{12} \propto \frac{q_1 q_2}{R_{12}^2}\hat{r}_{12}$$

$$\hookrightarrow \vec{F}_{12} = k\frac{q_1 q_2}{R_{12}^2}\hat{r}_{12}$$

constant of proportionality → depends on system of units. In SI (International system of Units)

$$k = \frac{1}{4\pi\epsilon_0};$$

$$\vec{R}_{12} = \vec{r}_1 - \vec{r}_2 = R_{12}\hat{r}_{12}$$

$$\epsilon_0 = 8.85\times10^{-12} \simeq \frac{10^{-9}}{36\pi}\ [F/m]$$

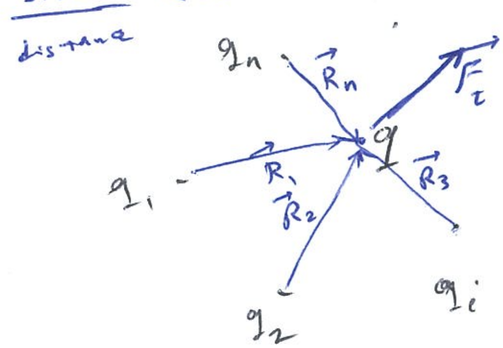
Permitivity of free space (vacuum).

$$\Rightarrow \vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2}\hat{r}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0}\cdot\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|^3}$$

→ for point charges or ~~bodies~~ charged bodies which can be considered as point charges.

$$\frac{\text{size}}{\text{distance}} \ll 1.$$

$$\vec{F}_{12} = -\vec{F}_{21}$$



Superposition principle:
$$\vec{F}_t = \sum_{i=1}^{n} q\,\frac{q_i(\vec{r}-\vec{r}_i)}{4\pi\epsilon_0|\vec{r}-\vec{r}_i|^3}$$

E-field intensity:

action-at-a-distance: a charge can exert force on another charge from a large distance.

Coloumb's law says if a charge moves toward another one the force must change instantaneously, but the theory of relativity says information about the motion of one charge will take some time to reach the other charge, since $v < c$.

→ There will be an unbalance in energy & momentum of the system of charges

relativity says: for interacting objects the momentum & energy can not be conserved by themselves. There must exist an extra entity, in the form of a perturbation in the medium in which the interacting bodies are situated, to account for the momentum & energy missing from the objects.

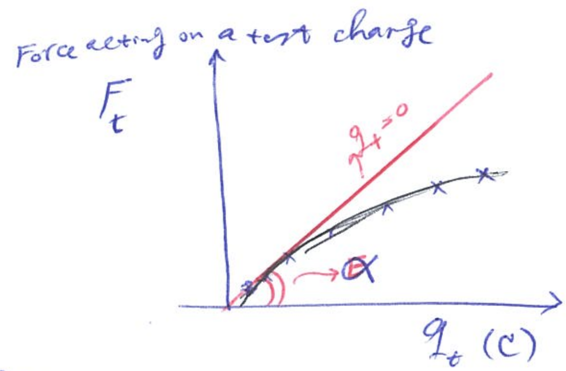
This extra entity is called the "Field".

⇒ Defining field will be useful to define the force acting on a charge in the presence of another charge.

⇒ "Electric Field" or "E.F. intensity": A property of space surrounding the charge. If any other charge comes in this space will experience a force. ⇒ action-by-contact.

Mesuring e.f. intensity: $+q_t$ test charge ⇒ E.F. intensity $= \frac{F}{q_t}$

But q_t makes its own E field and distorts the initial E.F. and experience force ⇒ we make q_t smaller & smaller and extrapolate the data to obtain the E.F. intensity in limit $q_t \to 0$



E = slope of the curve at $q_t \to 0$

$$\tan(\alpha) = \frac{F_t}{q_t} = E$$

$$\Rightarrow \boxed{\vec{E} = \lim_{q_t \to 0} \frac{\vec{F}}{q_t} \left[\frac{N}{C}\right]}$$

$\left[\frac{V}{m}\right] =$

$\vec{F}$ = total force acting on q_t.

$$\vec{F_q} = q\vec{E}$$ on space.