Electromagnetic Field Theory

* What is this course about?
  - What is a field?
  - How many types of fields exist? scalar / vector / continuous / rotational
  - What is the nature of a field?
  - How does a current-carrying coil produce a magnetic field?
  - How does a capacitor store energy?
  - How does a piece of wire (antenna) radiate or receive signals?
  - How do EM fields propagate in space?
  - How does EM energy travels from one end of a hollow pipe (waveguide) to the other?

* We will study the fundations of EM fields to answer these questions. ⇒ EM fields theory is vital to understanding many phenomena in nature and in electrical engineering.
* We explain the origin of equations in electrical engineering.

- what is a "field"?

In general, we define the behavior of a quantity in a region in terms of a set of values, one for each point in that region.

→ The behavior of ~~the~~ a quantity in a region is called a "field".

- The value of that quantity at each point of a field can be → measured experimentally
or → predicted by carrying out certain mathematical operations on some other quantities.

- Vector & scalar fields:
Both scalar & vector fields exist.

- Relations between field quantities:
Maxwell was able to calculate the speed of EM wave using the properties of medium:

ϵ = Permitivity
μ = Permeability

$$\rightarrow c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \ m/s$$

- Vector Analysis: is the language used in the study of EM fields.

- Gradiant, Divergence, & curl: we need to know vector operations to develope EM field theory.

Formulation of quantities in Integral of differential forms:

- $\int$ form: useful to explain the significance of an eqn.
- diff. form: useful for mathematical operations.

exp: eqn. of continuity of current:

$\vec{J}$ = volume current density

ρ = volume charge density

$$\begin{cases} \text{diff.:} & \vec{\nabla}\cdot\vec{J} = -\dfrac{\partial \rho}{\partial t} \\ \int: & \displaystyle\int_v \vec{\nabla}\cdot\vec{J}\, dv = -\int_v \frac{\partial \rho}{\partial t}\, dv \rightarrow \oint_S \vec{J}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_v \rho\, dv \end{cases}$$

diff: If the current density is known at a point, we can use this diff-form eqn. to calculate the rate at which the charge density is changing at that point.

$\int$: net outward current I through the closed surface $\underline{S}$ bounding volume v = the rate at which the charge inside the region is decreasing with time.

$$I = -\frac{dq}{dt}$$ :: circuit equation with a (−) sign.

prob.: write the integral form of $\vec{E} = -\vec{\nabla}V$.

$$V_{ba} = -\int_a^b \vec{E}\cdot d\vec{l}$$

EM static fields:

→ Electrostatics: assume
- all charges are fixed in space.
- charge densities are constant in time.
- charge is source of the E-field.

interest
- E-field intensity at any point
- the potential distribution
- the forces
- E-energy distribution in the region.
- how capacitor store energy?

Equations
- coulomb's law
- Gauss's law
- Poisson's eqn. } potential functions.
- Laplace's eqn. } potential functions.
- ohm's law

→ Magnetostatics: assume
- restricted charge movement → resulting current is constant in time.
  → This field is called magnetic field.
- current is constant in time → magnetic field is constant in time. ⟹ magnetostatics.

interest
- mag. field intensity
- mag. flux density
- mag. flux stored in the mag. field
- mag. energy " " " "

equations
- Biot-Savart law
- Amper's law
- Gauss's law for mag.

→ Applications of E & M static fields.

Time-varying EM fields :

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"E&M Field Theory"

Static E&M Fields: "constant in time" $\vec{F} = q\vec{E}$
{ static E-fields to accelerate a particle
{ ~ M- ~ ~ deflect ~ ~
⇒ - oscilloscope - Ink-Jet printer

↳ we need to study it throughly.

Time-varying Fields:

$emf(t) = -\frac{d\phi}{dt}$ Faraday's Law of induction $\xrightarrow{\text{maxwell}}$ Displacement current. → Prediction of propagation of fields in free space with the velocity of light.

n.b. ↳ "Transmission + Reception + propagation of energy."

⇓

$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ Lorentz Force equation.

→ Understanding EM theory: with { maxwell eqns, eqn of continuity, Lorentz force } → we can explain the effect of Electromagnetism.

→ solutions to maxwell's eqns: always leads to waves.

The nature of waves depend upon the medium, type of excitation (source) and boundary conditions.

⇓

$\boxed{\text{Light}}$

⇓

relativistic concepts

⇓

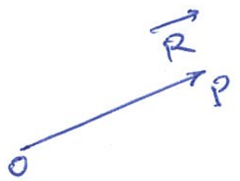
4-D equations → { gradient, divergence, curl, Laplacian } ⇒ "4D operator"

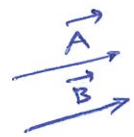
"Vector Analysis review"

"vector Algebra"

using vectors: they provide compact mathematical representations of complicated Phenomena, allow easy visualization & manipulation.

quantities in physics:
- scalar : completely described by its magnitude (mass, time, temperature, work, electric charge
- vector : magnitude + direction (Force, velocity, torque, electric field, accelera



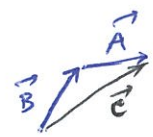


$\vec{A} = \vec{B}$ if they have the same magnitude & directio (length)

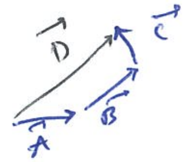
Null vector (zero vector) : $\vec{0}$

unit vector : $\vec{A} = A\hat{A}$; $\hat{A} = \frac{\vec{A}}{A}$

(+) vector addition: $\vec{A} + \vec{B} = \vec{C}$ → vector

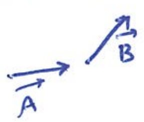


→ $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
"commutative law of addition"

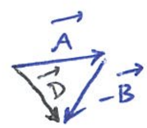


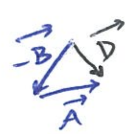
$\vec{D} = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) = \vec{A} + \vec{C} + \vec{B}$
"associative law of addition"

(−) vector subtraction:



$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$





$\vec{D} = \vec{A} + (-\vec{B})$
$= -\vec{B} + \vec{A}$

multiplication:

-multiplication of a vector by a scalar: $\vec{B} = k\vec{A}$
- $\vec{B}$ is in the same direction of $\vec{A}$ if $k>0$.
- $\vec{B}$ ~ in opposit ~ ~ if $k<0$.

$\begin{cases} |\vec{B}| > |\vec{A}| & \text{if } |k| > 1. \\ |\vec{B}| < |\vec{A}| & \text{if } |k| < 1 \end{cases}$ ⇒ $\vec{B}$ is always Parallel to $\vec{A}$ in direction or opposite.

→ $\vec{B}$ is called dependent vector.

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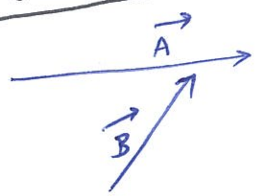
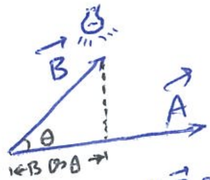
useful definitions:

Product of Vectors: $\begin{cases} \text{dot product } (\cdot) \text{ or scalar product.} \\ \text{cross } \text{"} \quad (\times) \end{cases}$

dot product:



$\vec{A}\cdot\vec{B} = (A)(\underbrace{B\cos\theta}_{\text{scalar.}})$; θ = smaller angle between $\vec{A}$ & $\vec{B}$.



$|B\cos\theta|$

if $\begin{cases} \theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow |\vec{A}\cdot\vec{B}| = \text{max.} \\ \theta = 90^\circ \Rightarrow \cos\theta = 0 \Rightarrow \vec{A}\cdot\vec{B} = 0 \rightarrow \text{"orthogonal vectors"} \end{cases}$

component of $\vec{B}$ along $\vec{A}$ ≡ scalar projection of $\vec{B}$ on $\vec{A}$.

✓ commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$; $\vec{A}\cdot\vec{B} \Rightarrow = A(B\cos\theta) = B(A\cos\theta) = \vec{B}\cdot\vec{A}$

✓ Distribute: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

✓ scaling: $k(\vec{A}\cdot\vec{B}) = (k\vec{A})\cdot\vec{B} = \vec{A}\cdot(k\vec{B})$

$\begin{cases} \text{scalar projection of } \vec{B} \text{ on } \vec{A}: \; B\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{A} = \vec{B}\cdot\hat{A} \\ \text{vector " of } \vec{B} \text{ on } \vec{A}: \; (B\cos\theta)\hat{A} = (\vec{B}\cdot\hat{A})\hat{A} = (\vec{B}\cdot\hat{A})\hat{A} \end{cases}$

$\vec{A}\cdot\vec{A} = A^2 \Rightarrow A = \sqrt{\vec{A}\cdot\vec{A}}$ → magnitude of vector $\vec{A}$.

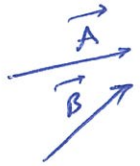
$\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$ $(\vec{A}\neq 0, \vec{B}\neq 0)$: angle between the two vectors $\vec{A}$ & $\vec{B}$. :Q

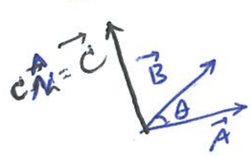
Ex1: if $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$, is it always $\vec{B} = \vec{C}$?

$\vec{A}\cdot\vec{B} - \vec{A}\cdot\vec{C} = 0 \Rightarrow \vec{A}\cdot(\vec{B}-\vec{C}) = 0$ $\begin{cases} \vec{A} = 0 \\ \vec{A} \perp (\vec{B}-\vec{C}) \\ \vec{B}-\vec{C} = 0 \Rightarrow \vec{B} = \vec{C} \text{ only.} \end{cases}$

cross product:

$\vec{A}\times\vec{B} = |AB\sin\theta|\, \underbrace{\hat{n}}_{\text{vector}}$ → unit vector normal to plane of $\vec{A}\times\vec{B}$.





$C\hat{n} = \vec{C}$

$\vec{C} = \vec{A}\times\vec{B}$

$C\hat{n} = (A\hat{A})\times(B\hat{B}) = AB(\hat{A}\times\hat{B}) = AB\sin\theta\,\hat{n}$

$\boxed{\hat{n} = \dfrac{\hat{A}\times\hat{B}}{|\sin\theta|}}$

→ $\hat{n} = \dfrac{\hat{A}\times\hat{B}}{|\sin\theta|}$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ → NO Commutative!

$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ Distributive ✓

$(k\vec{A})\times\vec{B} = k(\vec{A}\times\vec{B}) = \vec{A}\times(k\vec{B})$ scaling ✓

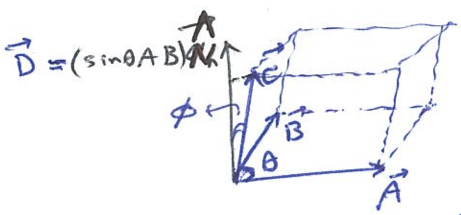
Q1: what is a necessary & sufficient condition for two non zero vectors to be parallel?

$$\vec{A} \parallel \vec{B} \rightarrow \theta = \begin{cases} 0 \\ \pi \end{cases} \rightarrow \sin\theta = 0 \Rightarrow AB\sin\theta = 0 \rightarrow \vec{A}\times\vec{B} = 0.$$

Ex: Lagrange's identity: $|\vec{A}\times\vec{B}|^2 = A^2B^2 - (\vec{A}\cdot\vec{B})^2$

$$|\vec{A}\times\vec{B}|^2 = A^2B^2\sin^2\theta = A^2B^2(1-\cos^2\theta) = A^2B^2 - A^2B^2\cos^2\theta = A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

– scalar triple product: $\vec{C}\cdot(\vec{A}\times\vec{B}) = ABC\sin\theta\cos\phi \rightarrow$ scalar



$\vec{D} = \vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

$C\cos\phi$ = projection in $\vec{A}\times\vec{B}$ direction.

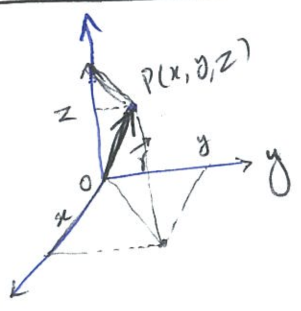
$\vec{C}\cdot\vec{D} = CAB\sin\theta\,\hat{C}\cdot\hat{n} = ABC\sin\theta\cos\phi$

the same: $\vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) =$ Volume

– vector triple product: $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$

Now: we need a coordinate system to calculate vectors numerically:

Coordinate systems:
- Rectangular c.s. (cartesian) (x, y, z)
- cylindrical c.s. (circular) (ρ, ϕ, z)
- spherical c.s. (r, θ, ϕ)



$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ Position vector.

$$\begin{cases} \vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \\ \vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \end{cases} \Rightarrow \vec{C} = \vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

$\vec{C} = C_x\hat{x} + C_y\hat{y} + C_z\hat{z}$

x

$$\begin{cases} \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1 \\ \hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0 \end{cases}$$

$$\begin{cases} \hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0 \\ \hat{x}\times\hat{y} = \hat{z},\ \hat{y}\times\hat{z} = \hat{x},\ \hat{z}\times\hat{x} = \hat{y} \end{cases}$$

$\hat{x} \rightarrow \hat{y} \rightarrow \hat{z} \rightarrow \hat{x}$

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

$A = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A^2 + A_y^2 + A_z^2}$

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$$\vec{C} = \vec{A} \times \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$= (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_x)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

⇒ calculate volume of a parallelepiped formed by $\vec{A}, \vec{B}, \vec{C}$?

$$V = \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Cylindrical coordinate system :

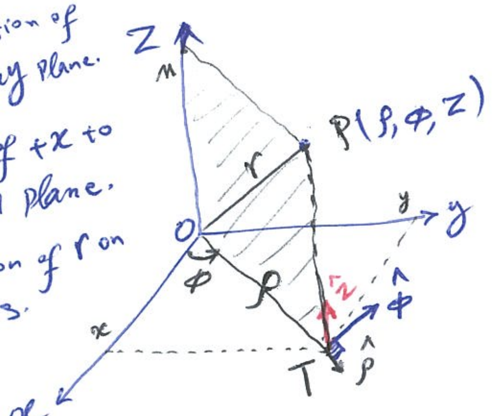
Three mutually perpendicular surfaces :

$$\begin{cases} z = \text{const.} \ \text{plane} \\ \rho = \text{constant} \ \text{cylinder} \\ \phi = \text{constant} \ \text{plane} \end{cases}$$

ρ = projection of r on xy plane.

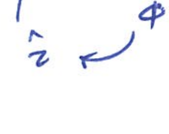
ϕ = angle of +x to OTPM plane.

z = projection of r on z axis.



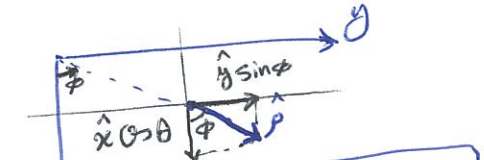
$$\begin{cases} \vec{A} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z} \\ \vec{B} = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z} \end{cases}$$

$$\begin{cases} \vec{A} + \vec{B} = (A_\rho + B_\rho)\hat{\rho} + (A_\phi + B_\phi)\hat{\phi} + (A_z + B_z)\hat{z} \\ \vec{A} \cdot \vec{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z \\ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \end{cases}$$

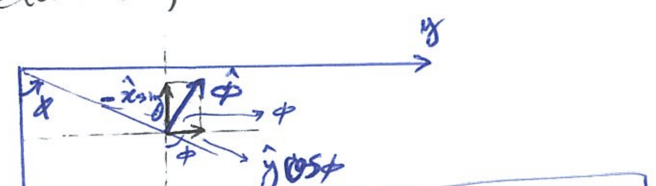


$$\begin{cases} \hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1 \\ \hat{\rho} \cdot \hat{\phi} = \hat{\rho} \cdot \hat{z} = \hat{\phi} \cdot \hat{z} = 0 \\ \hat{\rho} \times \hat{\rho} = \hat{\phi} \times \hat{\phi} = \hat{z} \times \hat{z} = 0 \\ \hat{\rho} \times \hat{\phi} = \hat{z}, \ \hat{\phi} \times \hat{z} = \hat{\rho} \\ \hat{z} \times \hat{\rho} = \hat{\phi} \end{cases}$$

Transformation of unit vectors : from rectangle to cylindrical c.s. :



$$\boxed{\hat{\rho} = \cos\theta\, \hat{x} + \sin\phi\, \hat{y}}$$



$$\boxed{\hat{\phi} = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y}}$$

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$$\begin{bmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

T : rectangular → cyl.

Transformation matrix

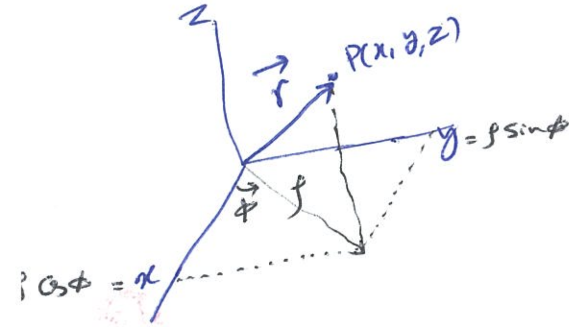
Transformation of a vector: from rect. to cyl:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

EX : Transforming Position vector at any Point in Space into a vector in cylindrical system:

$$\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$$

$[3\times3]\,[3\times1] = (3\times1)$



$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x\cos\phi + y\sin\phi + 0 \\ -x\sin\phi + y\cos\phi + 0 \\ 0 + 0 + z \end{bmatrix}$$

$$\Rightarrow \begin{cases} A_\rho = x\cos\phi + y\sin\phi \\ A_\phi = -x\sin\phi + y\cos\phi \\ A_z = z \end{cases} \Longrightarrow \begin{cases} A_\rho = \rho\cos^2\phi + \rho\sin^2\phi = \rho \\ A_\phi = -\rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi = 0 \\ A_z = z \end{cases}$$

$$\Rightarrow \boxed{\vec{A} = \rho\hat{\rho} + z\hat{z}}$$

Transforming a vector from cylindrical c.s. into Rectangular c.s.:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

* transformation of a vector from one c.s. to another does not change its magnitude or its direction!

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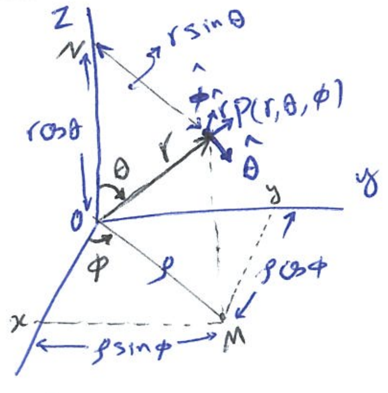
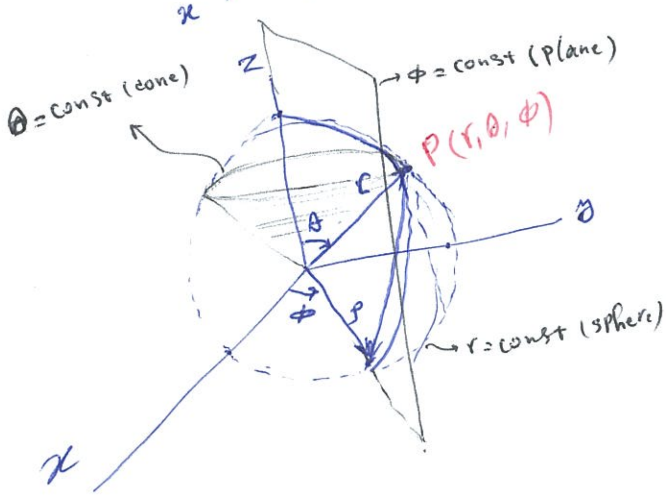
spherical coordinate system: (r, θ, ϕ)



$$\begin{cases} r = \text{radial distance from } O \text{ to } P. \ (\hat{r}) \\ \theta = \text{angle } r \text{ makes with } +z. \ (\hat{\theta}) \\ \phi = \text{angle between } +xz \text{ plane \& } OMPN. \ (\hat{\phi}) \end{cases}$$



$$\begin{cases} x = \rho\cos\phi = r\sin\theta\cos\phi \\ y = \rho\sin\phi = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\Downarrow$$

$$\begin{cases} r = (x^2+y^2+z^2)^{1/2} \Rightarrow (0 \le r \le \infty) \\ \theta = \cos^{-1}\left(\frac{z}{r}\right) \Rightarrow (0, \pi) \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow (0, 2\pi) \end{cases}$$

$$\begin{cases} r = \text{const (sphere)} \\ \theta = \text{const (cone)} \\ \phi = \text{const (plane)} \end{cases}$$

$\Rightarrow$ The tangent planes to this surfaces at $\underline{\underline{P}}$ are mutually perpendicular.

* $\hat{r}, \hat{\theta}, \hat{\phi}$ are functions of the coordinates (r, θ, ϕ) $\Rightarrow$ Vector (addition, subtraction & multiplicat of any two vectors in spherical coordinates can only be performed if these vectors are given at the intersection of θ = const & ϕ = const. planes.

$\Downarrow$

vectors must be defined either at the same point or at points along the same radial line.

$$\begin{cases} \hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1 \\ \hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{r} = 0 \\ \hat{r}\times\hat{\theta} = \hat{\phi}, \quad \hat{\theta}\times\hat{\phi} = \hat{r}, \quad \hat{\phi}\times\hat{r} = \hat{\theta} \end{cases}$$

$$\hat{r} \to \hat{\theta} \to \hat{\phi} \to \hat{r}$$

EX: find a unit vector perpendicular to both vectors $\vec{A}$ & $\vec{B}$.

$$\begin{cases} \vec{A} = 10\hat{r} + 30\hat{\theta} - 10\hat{\phi} \\ \vec{B} = -3\hat{r} - 10\hat{\theta} + 20\hat{\phi} \end{cases} \qquad \hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

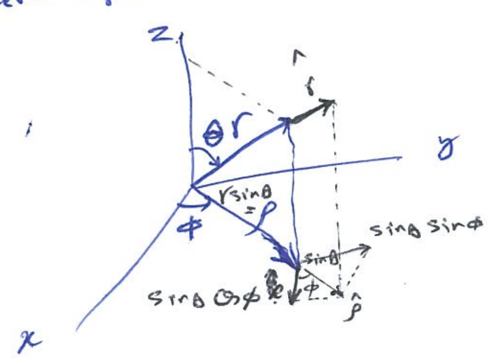
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 10 & 30 & -10 \\ -3 & -10 & 20 \end{vmatrix} = (600-100)\hat{r} - (200-30)\hat{\theta} + (-100+90)\hat{\phi}$$

$$= 500\hat{r} - 170\hat{\theta} - 10\hat{\phi}$$

$$|\vec{A}\times\vec{B}| = (500^2 + 170^2 + 10^2)^{1/2}$$

$$\Rightarrow \begin{cases} \hat{n}_1 = 0.947\hat{r} - 0.322\hat{\theta} - 0.01\hat{\phi} \\ \hat{n}_2 = -\hat{n}_1 \end{cases} \Rightarrow \text{Two unit vectors } \perp \text{ to } \vec{A} \times \vec{B}.$$

Transformation of unit vectors: If a set of vectors are given in spherical sys at different points but not along the same radial line (θ = const & ϕ = const.), we have to express them in rectangular system to perform vector operations:



$$\begin{cases} \hat{r}\cdot\hat{x} = \sin\theta\cos\phi \\ \hat{r}\cdot\hat{y} = \sin\theta\sin\phi \\ \hat{r}\cdot\hat{z} = \cos\theta \end{cases} \qquad \begin{cases} \hat{\theta}\cdot\hat{x} = \cos\theta\cos\phi \\ \hat{\theta}\cdot\hat{y} = \cos\theta\sin\phi \\ \hat{\theta}\cdot\hat{z} = -\sin\theta \end{cases}$$

$$\begin{cases} \hat{\phi}\cdot\hat{x} = -\sin\phi \\ \hat{\phi}\cdot\hat{y} = \cos\phi \\ \hat{\phi}\cdot\hat{z} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Tranforming a vector : $\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

we can obtain x-component of $\vec{A}$ by projecting it onto x-axis as:

$$A_x = \vec{A} \cdot \hat{x} = A_r \, \hat{r} \cdot \hat{x} + A_\theta \, \hat{\theta} \cdot \hat{x} + A_\phi \, \hat{\phi} \cdot \hat{x}$$

$$= A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \qquad (r,\theta,\phi \rightarrow x,y,z)$$

&

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad (x,y,z \rightarrow r,\theta,\phi)$$

*

EX : A vector $\vec{F} = 3x\,\hat{x} + 0.5y^2\,\hat{y} + 0.25x^2y^2\,\hat{z}$ is given at a Point $P(3,4,12)$ in the rectangular coordinate system. Express $\vec{F}$ in spherical c.s.:

$$\begin{cases} \vec{F}_p = 9\hat{x} + 8\hat{y} + 36\hat{z} \\ P(3,4,12) \,;\, r = (9+16+144)^{1/2} \,,\; r = 13 \end{cases} \quad ;\ \phi = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ \quad ;\ \theta = \cos^{-1}\left(\frac{12}{13}\right) = 22.62^\circ$$

*

$(x,y,z) \rightarrow (r,\theta,\phi)$

$$\Rightarrow \begin{cases} F_r = 37.77 \\ F_\theta = -2.95 \\ F_\phi = -2.40 \end{cases}$$

$$\Rightarrow \vec{F} = 37.77\,\hat{r} - 2.95\,\hat{\theta} - 2.40\,\hat{\phi}$$

at $P(13, 22.62^\circ, 53.13^\circ)$ in spherical c.s.