
DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING
COMP232 MATHEMATICS FOR COMPUTER SCIENCE
FALL 2017

Assignment 2. Solutions.

1. Let $P(x, y, z)$ denote the statement $x + y \leq z$, where $x, y, z \in \mathbb{Z}^+$. What is the truth value of each of the following? Explain your answers.

(a) $\forall x \exists y \exists z P(x, y, z)$.

Solution: True. Let x be any integer. By choosing $y = 1$ and $z = x + 1$, we see that the statement is true.

(b) $\forall y \exists x \forall z P(x, y, z)$.

Solution: False. If $y = 10$ and $z = 1$ then there is no x such that $10 + x \leq 1$

(c) $\exists z \exists y \forall x P(x, y, z)$.

Solution: False. x could be arbitrarily large.

2. For each of the premise-conclusion pairs below, give a valid step-by-step argument (proof) along with the name of the inference rule used in each step. For examples, see pages 73 and 74 in textbook.

(a) Premise: $\{\neg p \vee q \rightarrow r, s \vee \neg q, \neg t, p \rightarrow t, \neg p \wedge r \rightarrow \neg s\}$, conclusion: $\neg q$.

Solution:

Step	Conclusion	Reason
1.	$p \rightarrow t$	Premise
2.	$\neg t$	Premise
3.	$\neg p$	Modus Tollens using (1) and (2)
4.	$\neg p \vee q$	Addition using (3)
5.	$\neg p \vee q \rightarrow r$	Premise
6.	r	Modus Ponens using (4) and (5)
7.	$\neg p \wedge r$	Conjunction using (3) and (6)
8.	$\neg p \wedge r \rightarrow \neg s$	Premise
9.	$\neg s$	Modus Ponens using (7) and (8)
10.	$s \vee \neg q$	Premise
11.	$\neg q$	Disjunctive Syllogism using (9) and (10)

(b) Premise: $\{\neg p \rightarrow r \wedge \neg s, t \rightarrow s, u \rightarrow \neg p, \neg w, u \vee w\}$, conclusion: $\neg t \vee w$.

Solution:

Step	Conclusion	Reason
1.	$u \vee w$	Premise
2.	$\neg w$	Premise
3.	u	Disjunctive Syllogism using (1) and (2)
4.	$u \rightarrow \neg p$	Premise
5.	$\neg p$	Modus Ponens using (3) and (4)
6.	$\neg p \rightarrow r \wedge \neg s$	Premise
7.	$r \wedge \neg s$	Modus Ponens using (5) and (6)
8.	$\neg s$	Simplification using (7)
9.	$t \rightarrow s$	Premise
10.	$\neg t$	Modus Tollens using (8) and (9)
11.	$\neg t \vee w$	Addition using (10)

(c) Premise: $\{p \vee q, q \rightarrow r, p \wedge s \rightarrow t, \neg r, \neg q \rightarrow u \wedge s\}$, conclusion: t .

Solution:

Step	Conclusion	Reason
1.	$\neg r$	Premise
2.	$q \rightarrow r$	Premise
3.	$\neg q$	Modus Tollens using (1) and (2)
4.	$\neg q \rightarrow u \wedge s$	Premise
5.	$u \wedge s$	Modus Ponens using (3) and (4)
6.	s	Simplification using (5)
7.	$p \vee q$	Premise
8.	p	Disjunctive Syllogism using (3) and (7)
9.	$p \wedge s$	Conjunction using (6) and (8)
10.	$p \wedge s \rightarrow t$	Premise
11.	t	Modus Ponens using (9) and (10)

3. For each of the following, determine whether argument is valid. You may use a counterexample, equivalence transformations or truth-tables to justify your answer.

$$(a) \frac{p \rightarrow q}{\frac{\neg p}{\therefore \neg q}}$$

Solution: Invalid. Counterexample: p is *False* and q is *True*.

$$(b) \frac{\neg p \rightarrow \neg q}{\therefore (\neg p \rightarrow q) \rightarrow p}$$

Solution: Valid. We show using a truth-table that

$$\underbrace{(\neg p \rightarrow \neg q)}_a \rightarrow \underbrace{\left(\underbrace{(\neg p \rightarrow q)}_b \rightarrow p \right)}_c \text{ is a tautology.}$$

p	q	$\neg p$	$\neg q$	a $\neg p \rightarrow \neg q$	b $\neg p \rightarrow q$	c $(b \rightarrow p)$	$a \rightarrow c$
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>

$$(c) \frac{p \rightarrow r}{\frac{q \rightarrow r}{\frac{\neg(p \vee q)}{\therefore \neg r}}}$$

Solution: Invalid. Counterexample: p and q are *False* and r is *True*.

$$(d) \frac{p \rightarrow q}{\frac{p \rightarrow (q \rightarrow \neg p)}{\therefore \neg p}}$$

Solution: Valid. We show using a truth-table that

$$\left(\underbrace{(p \rightarrow q)}_a \wedge \underbrace{\left(p \rightarrow \underbrace{(q \rightarrow \neg p)}_b \right)}_c \right) \rightarrow \neg p \text{ is a tautology.}$$

p	q	$\neg p$	a $p \rightarrow q$	b $q \rightarrow \neg p$	c $p \rightarrow b$	$a \wedge c$	$a \wedge c \rightarrow \neg p$
<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>

4. For each of the arguments below, indicate whether it is valid or invalid.

- (a) All healthy people eat an apple a day.
Helen eats an apple a day.
 \therefore Helen is a healthy person.

Solution: Invalid. The argument is of the form

$$HP(\text{Helen}) \rightarrow EA(\text{Helen})$$

$$EA(\text{Helen})$$

$$\therefore HP(\text{Helen})$$

It is a **converse** error.

- (b) All healthy people eat an apple a day.
Herbert is not a healthy person.
 \therefore Herbert does not eat an apple a day.

Solution: Invalid. The argument is of the form

$$HP(\text{Herbert}) \rightarrow EA(\text{Herbert})$$

$$\neg HP(\text{Herbert})$$

$$\therefore \neg EA(\text{Herbert})$$

It is an **inverse** error.

- (c) If a product of two real numbers is 0, then at least one of the numbers is 0.
For a particular real number x , neither $(x - 1)$ nor $(x + 1)$ equals 0.
 \therefore The product $(x - 1)(x + 1)$ is not 0.

Solution: Valid. Let's use a instead of $x - 1$ and b instead of $x + 1$. The argument is of the form

$$\forall x \forall y (x \cdot y = 0 \rightarrow (x = 0 \vee y = 0))$$

$$a \neq 0 \wedge b \neq 0$$

$$\therefore a \cdot b \neq 0.$$

The argument is an instance of *Universal Modus Tollens*.

5. Use rules of inference to show that if $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$, and $\exists x(\neg R(x))$ are true, then $\exists x(\neg P(x))$ is true.

Solution:

- | | | |
|-----|------------------------------------|---|
| (1) | $\forall x(P(x) \rightarrow Q(x))$ | Premise |
| (2) | $P(c) \rightarrow Q(c)$ | Universal instantiation from (1) |
| (3) | $\forall x(Q(x) \rightarrow R(x))$ | Premise |
| (4) | $Q(c) \rightarrow R(c)$ | Universal instantiation from (3) |
| (5) | $P(c) \rightarrow R(c)$ | Hypothetical Syllogism from (2) and (4) |
| (6) | $\exists x(\neg R(x))$ | Premise |
| (7) | $\neg R(c)$ | Existential instantiation from (6) |
| (8) | $\neg P(c)$ | Modus Tollens from (5) and (7) |
| (6) | $\exists x(\neg P(x))$ | Existential Generalization from (7) |

6. (a) Give a direct proof of: “If x is an odd integer and y is an even integer, then $x + y$ is odd.”

Solution: If x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$, and if y is even, then $y = 2k'$ for some $k' \in \mathbb{Z}$. Consequently $x + y = 2k + 1 + 2k' = 2(k + k') + 1$ which means that $x + y$ is odd.

- (b) Give a proof by contradiction of: “If n is an odd integer, then n^2 is odd.”

Solution: Suppose n is odd and n^2 is even. Then $n = 2k + 1$, for some $k \in \mathbb{Z}$, and $n^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which contradicts the assumption that n^2 is even.

- (c) Give an indirect proof of: “If x is an odd integer, then $x + 2$ is odd.”

Solution: We show that if $x + 2$ is even, then x is even. If $x + 2$ is even, then $x + 2 = 2k$, for some $k \in \mathbb{Z}$. Therefore $x = 2k - 2 = 2(k - 1)$, which means that x is even.

- (d) Use a proof by cases to show that there are no solutions in positive integers to the equation $x^4 + y^4 = 100$.

Solution: The powers are $1^4 = 1$, $2^4 = 16$, $3^4 = 81$, $4^4 = 256$.

Therefore the possible cases for (x, y) where $x^4 + y^4 \leq 100$. are

- $(1, 1)$ which gives 2.
- $(1, 2)$ or $(2, 1)$ which gives 17.
- $(1, 3)$ or $(3, 1)$ which gives 82.
- $(2, 2)$ which gives 32.
- $(2, 3)$ or $(3, 2)$ which gives 97.

Consequently the equation $x^4 + y^4 = 100$ has no solutions in positive integers.

- (e) Prove that given a nonnegative integer n , there is a unique nonnegative integer m , such that $m^2 \leq n < (m + 1)^2$.

Solution:

Establish existence: Choose $m = \lfloor \sqrt{n} \rfloor$. Then $\sqrt{n} = m + \epsilon$, where $0 \leq \epsilon < 1$.

Thus $m^2 \leq (m + \epsilon)^2 = \sqrt{n}^2 = n = \sqrt{n}^2 = (m + \epsilon)^2 < (m + 1)^2$.

Show uniqueness: Suppose $p^2 \leq n < (p + 1)^2$ for some integer p . Then $p \leq \sqrt{n} < (p + 1)$, and since $\sqrt{n} = m + \epsilon$ we get $p \leq m + \epsilon < p + 1$, which, since $0 \leq \epsilon < 1$, is true only if $p = m$.

7. For each of the statements below state whether it is True or False. If True then give a proof. If False then explain why, e.g., by giving a counterexample.

(a) The difference of any two odd integers is odd.

Solution: The claim is false. Counterexample: $n = 7, m = 3, m - n = 4$.

(b) Let a and b be integers. If $a + b$ is even, then either a or b is even.

Solution: The claim is false. Counterexample: $1 + 1 = 2$.

(c) For all positive integers n , it holds that n is even if and only if $3n^2 + 8$ is even.

Solution: Only if direction: Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Consequently $3n^2 + 8 = 3(2k)^2 + 8 = 12k^2 + 8 = 2(6k^2 + 4)$ which means that $3n^2 + 8$ is even.

If direction: Proof by contraposition: Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Consequently $3n^2 + 8 = 3(2k + 1)^2 + 8 = 3(4k^2 + 4k + 1) + 8 = 12k^2 + 12k + 3 + 8 = 2(6k^2 + 6k) + 10 + 1 = 2(6k^2 + 6k + 5) + 1$, which means that $3n^2 + 8$ is odd.

(d) For all positive $x, y \in \mathbb{R}$, if x is irrational and y is irrational then $x + y$ is irrational.

Solution: The statement is false. We first show that $2 + \sqrt{2}$ is irrational. Suppose it is not. Then $2 + \sqrt{2} = \frac{a}{b}$, for some integers a and b , with $b \neq 0$. But then $\sqrt{2} = \frac{a}{b} - 2 = \frac{a - 2b}{b}$ which would mean that $\sqrt{2}$ is rational; a contradiction since we already have shown that $\sqrt{2}$ is irrational. Similarly we see that $2 - \sqrt{2}$ is irrational. We now have a counter-example, since $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$, which is rational. Note that $2 - \sqrt{2}$ is positive.

(e) $\forall x, y \in \mathbb{R}$, if x is irrational and y is rational then xy is irrational.

Solution: The statement is false. Counterexample: Let $x = \sqrt{2}$, which is irrational, and $y = 0$, which is rational. Then $xy = 0$, which is rational.