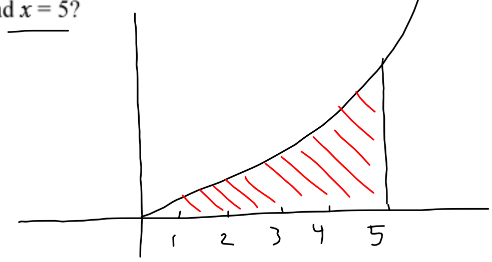
Chapter 5 – Integrals

5.1 – Calculating Areas using Riemann Sums

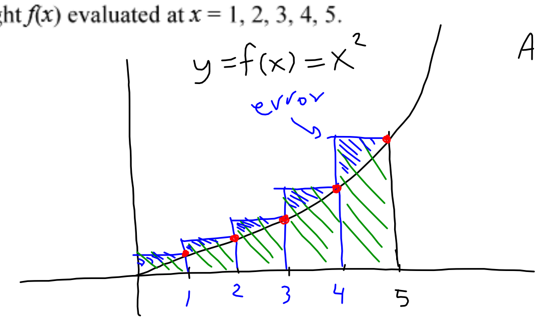
We have "nice" formulas for calculating the area of a circle, rectangle, parallelogram, triangle, etc. What about the area under the curve $f(x) = x^2$, above the x-axis between $x = 0$ and $x = 5$?



We can approximate this area by the sum of the areas of the five rectangles with width 1 and height $f(x)$ evaluated at $x = 1, 2, 3, 4, 5$.

0 5

$[0,1)$
$[1,2)$
$[2,3)$
$[3,4)$
$[4,5)$



$A \le (1)\, f(1) + (1)\, f(2) + (1)\, f(3) + (1)\, f(4) + (1)\, f(5)$

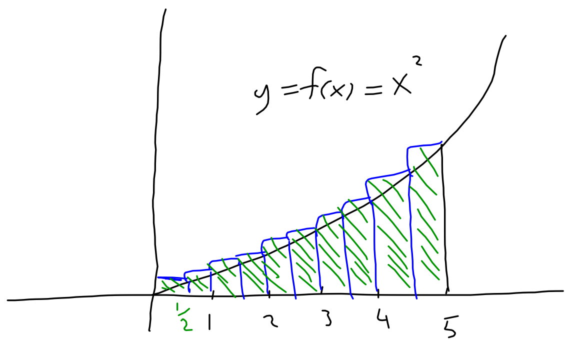
$= 1(1)^2 + 4 + 3^2 + 4^2 + 5^2$

$= 55$

$A \le 55$

This is not a great approximation, however, if we use the ten rectangles of width $\frac{1}{2}$, we get a better approximation. $x = 0.5, 1, 1.5, 2, 2.5, 3, \ldots, 5$

$[0, \frac{1}{2})$
$[\frac{1}{2}, 1)$
$[1, 1.5)$
$\vdots$
$[4.5, 5)$



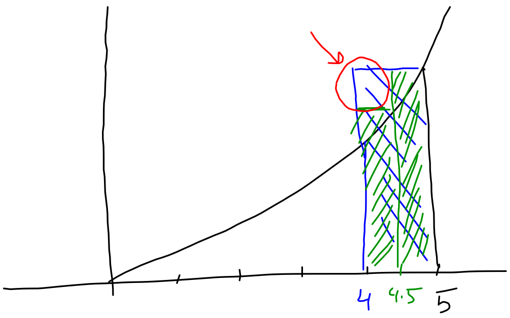
$A \le (\frac{1}{2})\, f(\frac{1}{2}) + \frac{1}{2} f(1) + (\frac{1}{2})\, f(1.5) + \frac{1}{2} f(2) + \frac{1}{2} f(2.5) + \cdots + \frac{1}{2} f(5)$

$= 48.125$

The exact value is $41.\overline{6}$

The more rectangles we take, the better the approximation is.

height of rectangle $= f(\text{right endpoint})$

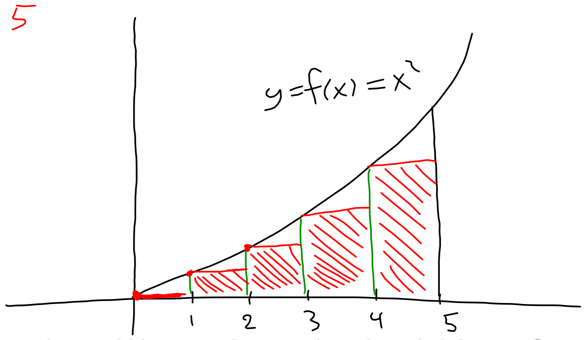


Notice that our estimates in the last example were too large. These are called <u>upper estimates</u>; we could have estimated the area by using <u>lower estimates</u>:

$x = 0, 1, 2, 3, 4, 5$

$\Delta x = 1$

$[0,1]$
$[1,2]$
$[2,3]$
$[3,4]$
$[4,5]$



$A \geq (1)\,f(0) + (1)\,f(1) + (1)\,f(2) + (1)\,f(3) + (1)\,f(4)$

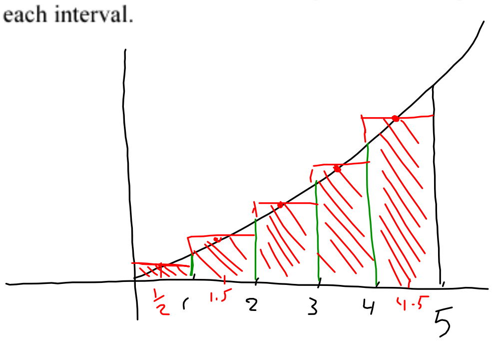
$= 30$

$41.\overline{6}$

We also could have used rectangles whose heights are $f(x)$ evaluated at the <u>midpoint</u> of each interval.

of $[a,b]$ is $\frac{a+b}{2}$

$[0,1]$
$[1,2]$
$\vdots$
$[4,5]$



$A \approx (1)\,f(\tfrac{1}{2}) + (1)\,f(1.5) + (1)\,f(2.5) + f(3.5) + (1)\,f(4.5)$

$= 41.25$

Sigma Notation and Limits of Finite Sums

Def'n: Sigma notation enables us to write a sum with many terms in a compact form.

$$\sum_{k=1}^{n} a_k$$

Σ stands for "sum".

Ex5.1) 1. $\sum_{k=1}^{5} k = 1+2+3+4+5 = 15$

2. $\sum_{k=2}^{6} k^2 = 2^2+3^2+4^2+5^2+6^2 = 90$

3. $\sum_{k=1}^{5} (-1)^k = (-1)^1+(-1)^2+(-1)^3+(-1)^4+(-1)^5$
$= \cancel{-1}\cancel{+1}\cancel{-1}\cancel{+1}-1 = -1$

4. $\sum_{k=10}^{12} (-1)^k \frac{k}{k+1} = (-1)^{10}\frac{10}{10+1} + (-1)^{11}\frac{11}{11+1} + (-1)^{12}\frac{12}{12+1} = \frac{10}{11} - \frac{11}{12} + \frac{12}{13}$
≈ 0.9155

Properties of Sigma Notation

1. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \Rightarrow$ $ca_1 + ca_2 + \cdots + ca_n = c(a_1+a_2+\cdots+a_n)$
$= c\sum_{k=1}^{n} a_k$

4. $\sum_{k=1}^{n} c = nc \Rightarrow$ $\underbrace{c+c+c+\cdots+c}_{n \text{ times}} = nc$

5. $\sum_{k=1}^{n} 1 = n$

$$\sum_{k=1}^{100} k = \frac{100(100+1)}{2}$$

Some Helpful Formulas:

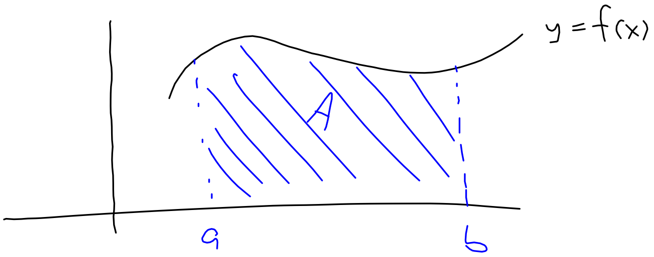
1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Ex: $\displaystyle\sum_{k=1}^{5} k = \frac{5(5+1)}{2} = 15$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

We can use all of this information (and limits) to help calculate the area under the graph of a function $f(x)$, above the x-axis and between two values $x = a$ and $x = b$.



$x_1 = a + 1\Delta x$

$x_2 = a + 2\Delta x$

$\vdots$

$x_k = a + k\Delta x$

We estimate this area by the sum of the areas of the rectangles of equal width Δx.
If there are n rectangles, then $\Delta x = \dfrac{b-a}{n}$. The height of the k^{th} rectangle is $f(x_k)$, where $x_k = a + k(\Delta x)$. That is, x_k is the right endpoint of the k^{th} interval.

So, $A \approx f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \displaystyle\sum_{k=1}^{n} f(x_k)\Delta x$

But this approximation becomes better as the number of rectangles increases. We get the exact area when the number of rectangles goes to infinity. That is, when $n \to \infty$.

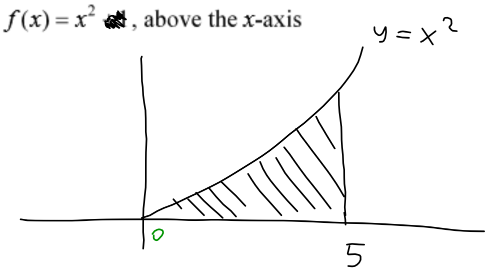
If $f(x) > 0$ on all $a \le x \le b$, then the area of the region under the graph of $f(x)$, above the x-axis and between $x = a$ and $x = b$ is given by:

$$A = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \quad (*)$$

where $\begin{cases} \Delta x = \dfrac{b-a}{n} \\ x_k = a + k\Delta x \end{cases}$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

Ex5.2) Calculate the area of the region under the graph of $f(x) = x^2$, above the x-axis and between $x = 0$ and $x = 5$.



$a = 0 \qquad b = 5$

$$\Delta x = \frac{b-a}{n} = \frac{5}{n}$$

$$x_k = a + k\Delta x = k\frac{5}{n}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(5\frac{k}{n}\right)^2\left(\frac{5}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} 5^3\frac{k^2}{n^3} = \lim_{n\to\infty} \frac{5^3}{n^3}\left[\sum_{k=1}^{n} k^2\right] = \frac{n(n+1)(2n+1)}{6}$$

$$= 5^3 \lim_{n\to\infty} \frac{1}{n^{\cancel{3}\,2}}\,\frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{5^3}{6} \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = \frac{5^3}{6}\cdot 2 = \frac{125}{3} = 41.\overline{6}$$

units squared

5.2 – The Definite Integral

Recall, we have used an infinite sum to calculate the area under a curve, above the x-axis and between the values $x = a$ and $x = b$.

We use the notation $A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$ to denote this.

Def: The definite integral of f from a to b is

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x .}$$

provided that the limit exists and independent of the choice of the sample points x_k.

Def'n: A function is integrable on $[a, b]$ if $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$ exists (is finite).

Theorem: A function that is continuous on $[a, b]$ is integrable on $[a, b]$.

Remark: Non-continuous functions may or may not be integrable.

Theorem: If f has only a finite number of jump discontinuities on $[a,b]$, then f is integrable on $[a,b]$

Ex: Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin(x_i)) \Delta x$ $= \int_0^{\pi} (x^3 + x\sin(x))\, dx$ as an integral on the interval $[0, \pi]$

$a = 0$, $b = \pi$

$\int \equiv$ sum

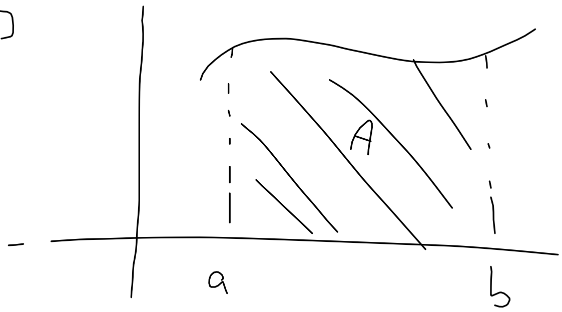
$\lim \sum \longrightarrow \int$

$x_i \longrightarrow x$

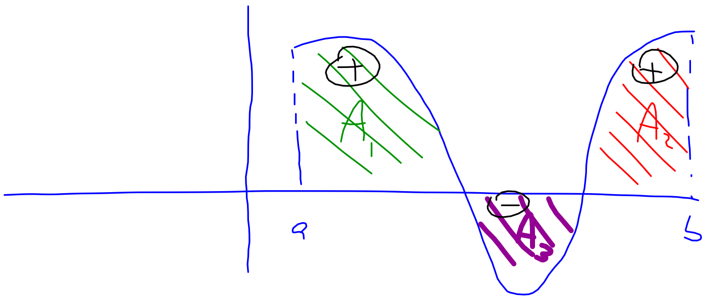
$\Delta x \longrightarrow dx$

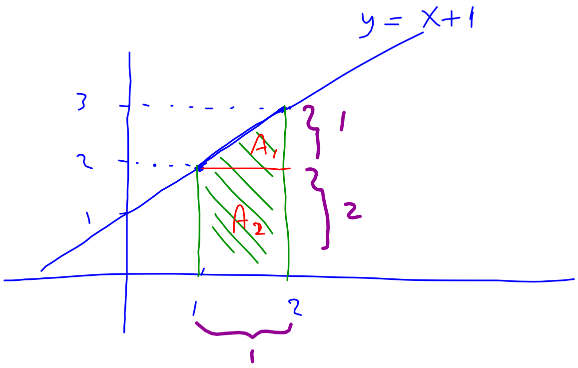
If $f(x) \geq 0$ for all $x \in [a,b]$

$$A = \int_a^b f(x)\,dx$$



$$\int_a^b f(x)\,dx = \text{"Net area"}$$
$$= A_1 + A_2 - A_3$$





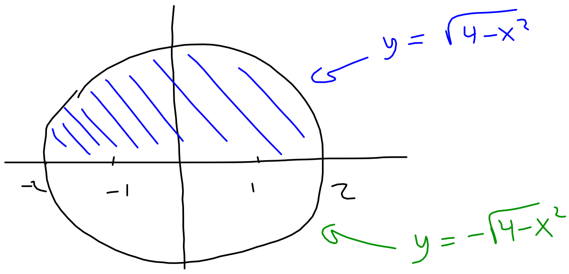
Ex5.3) Calculate $\int_{1}^{2}(x+1)dx$.

$$\int_{1}^{2}(x+1)\,dx = A_1 + A_2$$
$$= \tfrac{1}{2}(1)(1) + (1)(2)$$
$$= 2.5$$

$$y = +\sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2$$
$$x^2 + y^2 = 4$$

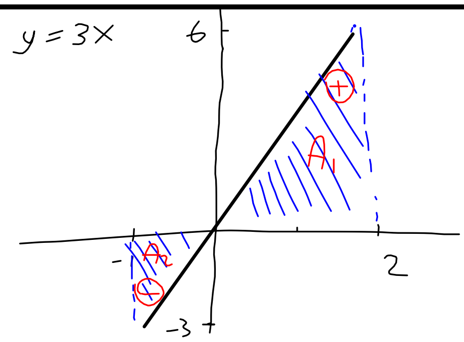
Ex5.4) Calculate $\int_{-2}^{2}\sqrt{4-x^2}\,dx$.

$$\int_{-2}^{2}\sqrt{4-x^2}\,dx = \tfrac{1}{2}\text{ Area of the circle}$$
$$= \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(2)^2$$
$$= 2\pi$$



Ex5.5) Calculate $\int_{-1}^{2}3x\,dx$.

$$\int_{-1}^{2}3x\,dx = A_1 - A_2$$
$$= \tfrac{1}{2}(2)(6) - \tfrac{1}{2}(1)(3)$$
$$= 6 - 1.5 = 4.5$$

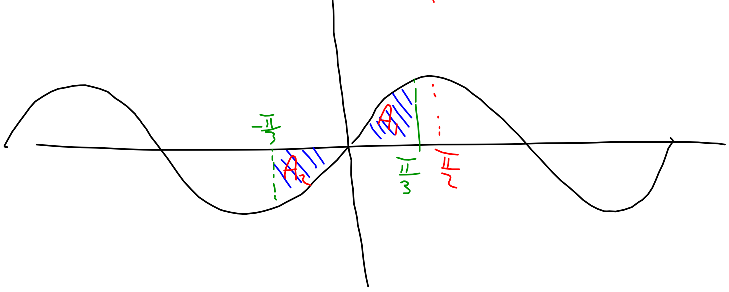


Ex5.6) Calculate $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sin x\,dx$.

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sin(x)\,dx = A_1 - A_2 = 0$$

since $A_1 = A_2$

$A_1 = A_2$



Properties of Definite Integrals:

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, where k is any constant

4. $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

* 5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

!! * Remark: $\int_a^b \left(f(x) \cdot g(x)\right)dx \neq \int_a^b f(x)dx \cdot \int_a^b g(x)dx$

6. $\int_a^b c\, dx = c(b-a)$