Integrals by Substitution

Review: Antiderivative Rules

Def: We say F is an *antiderivative* for f if $F'(x) = f(x)$.

If $f(x)$ is…	…then an antiderivative is…
x^n	$\frac{1}{n+1} x^{n+1}$ **except if** $n = -1$
1	x (assuming the variable is x!)
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
e^x	e^x
$\frac{1}{x}$	$\ln\lvert x\rvert$ (makes it work for $x < 0$ too.)

Antiderivatives are linear. So if F is an antiderivative for f and G is an antiderivative for g, then an antiderivative for $b\,f(x) + c\,g(x)$ is $b\,F(x) + c\,G(x)$

Review: The Indefinite Integral

We will use the notation $\int f(x)\,dx$

to represent *all possible* antiderivatives of the function $f(x)$, with respect to the variable x.

Called the ***indefinite integral*** of $f(x)$.

The Chain Rule

Recall: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$. For example: $\frac{d}{dx}\sin(x^2) = \cos(x^2)2x$

So $\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$

Looks almost like $\cos(x^2)\ 2x$, which is the derivative of $\sin(x^2)$.

Example: Evaluate: $\int x\cos(x^2)\, dx$

We will rearrange the integral to get an exact match:

$$\int x\cos(x^2)\, dx = \int \cos(x^2)\, x\, dx$$

$$= \tfrac{1}{2}\int \cos(x^2)\, 2\, x\, dx$$

$$= \tfrac{1}{2}\left[\sin(x^2)\right] + C$$

We put in a 2 so the pattern will match.

So we must also put in a 1/2 to keep the problem the same.

Check: $\frac{d}{dx}\left(\tfrac{1}{2}\sin(x^2)\right) = \tfrac{1}{2}\cos(x^2)2x = x\cos(x^2)$

From the chain rule

Integrals by Substitution

Start with $\int f(g(x))\, g'(x)\, dx$

Let $u = g(x)$. So we get:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, \underbrace{g'(x)\, dx}_{du} = \int f(u)\, du$$

Now need antiderivative of f, with u plugged in.

Rewrite the integral using the fact that

$$\frac{du}{dx} = g'(x) \qquad \text{so} \qquad du = g'(x)\, dx$$

Integrals by Substitution

Suppose we are trying to find $\int (3x^2+1)^3 x\,dx$

Here, the inside function is $u = 3x^2 + 1$.

So $du/dx = 6x$, or $du = 6x\,dx$.

Substitute:

$$\int (3x^2+1)^3 x\,dx = \int u^3 x\,dx$$

$$= \tfrac{1}{6}\int u^3 6x\,dx$$

$$= \tfrac{1}{6}\int u^3 du = \tfrac{1}{6}\left(\tfrac{1}{4}u^4\right) + C$$

$$= \tfrac{1}{24}(3x^2+1)^4 + C$$

Put in the 6 we need to get *du*

Have to cancel the 6 we put in

Our question wasn't about *u*!

(Use the fact that $u = 3x^2 + 1$.)

Integrals by Substitution

1) Choose u.
2) Calculate du. $$du = \frac{du}{dx} dx$$
3) Substitute u.
   Arrange to have du in your integral also.
   (**All *x*s and *dx*s must be replaced**!)
4) Solve the new integral.
5) Substitute back in to get x again.

Example: A linear substitution: $\int e^{3x+2} dx$

Let $u = 3x + 2$. Then $du = 3dx$.

$$\int e^{3x+2} dx = \int e^u dx = \frac{1}{3} \int e^u 3dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x+2} + C$$

A Second Approach

Instead of rearranging your integral to produce a *du*, you can solve for the differential *dx* (or *dt*, or whatever) and substitute for the differential.

You may need to simplify, and **you *still* must end up with only *u* or *du* and no *x* or *dx*.**

Example: $\int \cos(5x)\,dx$

Solve for *dx* here

Let $u = 5x$ so $du = 5dx$, or $dx = du/5$.

$$\int \cos(5x)\,dx = \int \cos(u)\frac{du}{5} = \frac{1}{5}\int \cos(u)\,du = \frac{1}{5}\sin(u) + C = \frac{1}{5}\sin(5x) + C$$

Replace *dx* with what it's equal to

Note: This is the same as the book's rule for cos(*kx*).

Choosing u

- Try to choose u to be an inside function. (Think chain rule.)
- Try to choose u so that du is in the problem, except for a constant multiple.

Example1: For $\int (3x^2+1)^3 x\,dx$

$u = 3x^2 + 1$ was a good choice because

(1) $3x^2 + 1$ is inside the cube.

(2) The derivative is $6x$, and we have an x.

Example2: For $\int e^{3x+2}dx$

$u = 3x + 2$ was a good choice because

(1) $3x + 2$ is inside the exponential.

(2) The derivative is 3, which is *only* a constant.

Practice

$$\int \frac{x}{x^2+1}\,dx$$

Let $u = x^2 + 1$ $du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \tfrac{1}{2}\int \frac{1}{x^2+1}\,2x\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+1\right| + C$$

$$\int \sin(7x+2)\,dx$$

Let $u = ?$

Let $u = 7x + 2$ So $du = 7\,dx$, and

$$\int \sin(7x+2)\,dx = \frac{1}{7}\int \sin(7x+2)\,7\,dx = \frac{1}{7}\int \sin u\,du = -\frac{1}{7}\cos(7x+2) + C$$

Practice

$\int \sin(x)\cos(x)dx.$ (Hint: There are several ways to do this.)

Let $u = ?$

Method 1

$$u = \sin(x) \quad \Rightarrow du = \cos(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \int u\,du$$

$$= \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C$$

Method 2

$$u = \cos(x) \Rightarrow du = -\sin(x)dx$$

$$-1\int \cos(x)\left(-\sin(x)\right)dx = -\int u\,du$$

$$-\frac{u^2}{2} + C = -\frac{\cos^2(x)}{2} + C$$

What's the difference?

This is 1!

$$\left(\tfrac{1}{2}\sin^2(x)\right) - \left(-\tfrac{1}{2}\cos^2(x)\right) = \tfrac{1}{2}\left(\sin^2(x) + \cos^2(x)\right) = \tfrac{1}{2}$$

That is, the difference is a *constant*.

Practice

$$\int \frac{e^x}{1+e^x}dx$$ **Let $u = ?$** $u = 1 + e^x \quad \Rightarrow du = e^x dx$

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln\left|1+e^x\right| + C$$

$$\int x\sin\left(x^2\right)dx$$ **Let $u = ?$** $u = x^2 \Rightarrow du = 2x$

$$\int x\sin(x^2)dx = \frac{1}{2}\int \sin(x^2)\,2x\,dx = \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(x^2) + C$$

Doesn't Fit All

1- We can't use u–substitution to solve everything. For example:

$$\int \sin(x^2)\,dx = \int \sin(u)\,dx$$
$$= \int \sin(u)\frac{du}{2x}$$
$$= \int \sin(u)\frac{1}{2x}\,du$$

Let $u = x^2$
$du = 2x\,dx$

We need $2x$ this time, not just 2.

We CANNOT multiply by a variable to adjust our integral.

We cannot complete this problem

2- For the same reason, we can't do the following by u–substitution:

$$\int (x^2 - 3)^3\,dx$$

But we already knew how to do this!

$$\int (x^2 - 3)^3\,dx = \int x^6 - 9x^4 + 27x^2 - 27\,dx = \frac{1}{7}x^7 - \frac{9}{5}x^5 + 9x^3 - 27x + C$$

Morals

- No one technique works for everything.
- Don’t forget things we already know!

Bad News

There are lots of integrals we will *never* learn how to solve…

In fact, there are a number of functions without elementary antiderivatives. That’s why numerical integrations are useful!

problem

Last time, we determined that

$$\int (3x+1)^2 dx = 3x^3 + 3x^2 + x + C$$

Now use *u*–substitution to compute the same integral.

Do you get the same result? (Don't just *assume* or *claim* you do; multiply out your result to *show* it!)

If you don't get exactly the same answer, is it a problem? Why or why not?

Solution

Last time, we determined that

$$\int (3x+1)^2 dx = 3x^3 + 3x^2 + x + C$$

$u = 3x+1$

$du = 3dx$

$$\int (3x+1)^2 dx = \frac{1}{3}\int (3x+1)^2 3dx = \frac{1}{3}\int u^2 du = \frac{1}{3}\frac{u^3}{3} + C$$

$$= \frac{1}{9}(3x+1)^3 + C$$

Are these answers equal?

$$\frac{1}{9}(3x+1)^3 + C = \frac{1}{9}(27x^3 + 27x^2 + 9x + 1) + C = 3x^3 + 3x^2 + x + \frac{1}{9} + C$$

$$= 3x^3 + 3x^2 + x + C_1$$

Yes. They may differ by a constant.

Summary

- Make a *u*-substitution: find *u* and *du*, then transform to an integral we can do. Be sure to transform back to the original variable!
- Rearrange to get *du*
- Substitution will *still* not solve every integral. (Nor is it always needed!)

Antiderivative Practice

Problem 1 $\int 2\sin(3t) - e^{4t} + 4^t\,dt$ Use basic formulas:

$$\int 2\sin(3t) - e^{4t}dt = -\tfrac{2}{3}\cos(3t) - \tfrac{1}{4}e^{4t} + \tfrac{1}{\ln(4)}4^t + C$$

Problem 2 $\int \frac{z^2+2z}{z^2}dz$ Simplify algebraically first, then integrate.

$$\int \frac{z^2+2z}{z^2}dz = \int \frac{z^2}{z^2} + \frac{2z}{z^2}dz = \int 1 + \frac{2}{z}dz = z + 2\ln|z| + C$$

Problem 3 $\int \frac{6t}{4+t^2}dt$ Make a substitution: Let u=4+t^2, so du=2tdt.

$$\int \frac{6t}{4+t^2}dt = 3\int \frac{2t}{4+t^2}dt = 3\int \frac{1}{u}du = 3\ln|u| + C = 3\ln\left|4+t^2\right| + C$$

Antiderivative Practice

Problem 4 $\int \frac{2}{y\ln(ky)}dy$ Make a substitution: u=ln(ky), so du=dy/y.

$$\int \frac{2}{y\ln(ky)}dy = \int \frac{2}{\ln(ky)}\frac{1}{y}dy = \int \frac{2}{u}du = 2\ln|u| + C = 2\ln|\ln(ky)| + C$$

Problem 5 Find the particular function $F(x)$ such that $F'(x) = x^2$ and the graph of $F(x)$ passes through (1, 2).

The general antiderivative is $\int x^2 dx = \frac{1}{3}x^3 + C$

Then to find C, we must have $F(1) = \frac{1}{3}1^3 + C = 2$

Thus, C = 5/3, and our function is $F(x) = \frac{1}{3}x^3 + \frac{5}{3}$