## 3. INTEGRALS

a) An object on the $x$-axis has velocity $v(t) = 2t - t^2$ at tme $t$. If it starts out at $x = -1$ at time $t = 0$, where is it at time $t = 3$? How far has it travelled?

**Solution:** Let $x = F(t)$ be the position of the object at time $t$. Its displacement from $t = 0$ to $t = 3$ is

$$\int_0^3 2t - t^2 \, dt = \left( t^2 - \frac{t^3}{3} \right) |_0^3 = 0.$$

This means that at $t = 3$ the object is again at $x = -1$.

In order to answer how far it has travelled, notice that the object turns around as $v$ changes sign. In other words, when $v(t) = 0$ is when the object changes direction: for $0 < t < 2$ we have that $v(t) > 0$, and for $2 < t < 3$ we have that $v(t) < 0$. Thus, the total distance travelled is

$$\int_0^2 2t - t^2 \, dt - \int_2^3 2t - t^2 \, dt = \frac{8}{3}.$$

b) Obtain $\displaystyle \frac{d}{dx}\int_a^{\sin(x)} \arctan(\ln(t)-1)\,dt.$

**Solution:**

$$\frac{d}{dx}\int_a^{\sin(x)} \arctan(\ln(t)-1)\,dt = \arctan\left(\ln(\sin(x))-1\right)\cos(x).$$

c) Find $f(x)$ if $f'(x) = xe^x$ and $f(0) = 1$.

**Solution:** Since

$$\int xe^x\,dx = xe^x - e^x + C$$

Use integration by parts IPP

we have that

$$f(x) = xe^x - e^x + C$$

Now, from

$$1 = f(0) = 0e^0 - e^0 + C = -1 + C$$

we conclude

$$f(x) = xe^x - e^x + 2.$$

d) Find

(i) $\displaystyle \int \sin^7(x)\,dx.$

**Solution:** Particular Case, Trigonometric Function: First

$$\int \sin^7(x)\,dx = \int \sin(x)\sin^6(x)\,dx = \int \sin(x)(1-\cos^2(x))^3\,dx$$

**Take u= cos(x)**

(ii) $\displaystyle\int \frac{x^3}{\sqrt{x^4+2}}\,dx.$

**Solution:** Proceed by Substitution: take $u = x^4 + 2$, so $du = 4x^3\,dx$. Then

$$\int \frac{x^3}{\sqrt{x^4+2}}\,dx = \frac{1}{4}\int u^{-1/2}\,du = \frac{\sqrt{x^4+2}}{2} + C.$$

(iii) $\displaystyle\int \frac{\sin(x)\,\ln(\cos(x))}{\cos(x)}\,dx.$

**Solution:** Proceed by Substitution: Take $u = \ln(\cos(x))$, so $du = \dfrac{-\sin(x)}{\cos(x)}\,dx$. Then

$$\int \frac{\sin(x)\,\ln(\cos(x))}{\cos(x)}\,dx = -\int u\,du = -\frac{\ln(\cos(x))}{2} + C.$$

(iv) $\displaystyle\int x^2\,e^{5x}\,dx.$

**Solution:** Proceed by Parts: First take $u = x^2$, $dv = e^{5x}\,dx$, so we have $du = 2x\,dx$ and $v = e^{5x}/5$. Then

$$\int x^2 e^{5x}\,dx = \frac{x^2e^{5x}}{5} - \frac{2}{5}\int xe^{5x}\,dx.$$

Again by parts, with $u = x$, $dv = e^{5x}\,dx$, so we have $du = dx$ and $v = e^{5x}/5$, we have now that

$$\int xe^{5x}\,dx = \frac{xe^{5x}}{5} - \int \frac{e^{5x}}{5}\,dx = \frac{xe^{5x}}{5} - \frac{e^{5x}}{5^2}.$$

Putting all together we obtain

$$\int x^2e^{5x}\,dx = \frac{x^2e^{5x}}{5} - \frac{2xe^{5x}}{5^2} + \frac{2e^{5x}}{5^3} + C.$$

(v) $\displaystyle\int x\sin(3x)\,dx.$

**Solution:** Proceed by Parts: Take $u = x$ and $dv = \sin(3x)\,dx$, so we have $du = dx$ and $v = -\cos(3x)/3$. Then

$$\int x\sin(3x)\,dx = -\frac{x\cos(3x)}{3} + \frac{1}{3}\int \cos(3x)\,dx = -\frac{x\cos(3x)}{3} - \frac{\sin(3x)}{3^2} + C.$$

(vi) $\displaystyle\int \frac{x-1}{x^3+x^2}\,dx.$

**Solution:** Special Case, Partial Fractions: Since

$$\frac{x-1}{x^3+x^2} = \frac{x-1}{x^2(x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} = \frac{A_1x(x+1)+A_2(x+1)+A_3x^2}{x^2(x+1)}$$

we should have

$$x-1 = A_1x(x+1)+A_2(x+1)+A_3x^2$$

Taking $x=0$ we obtain

$$-1 = A_2.$$

Taking $x=1$ we obtain

$$-2 = A_3.$$

Thus,

$$x-1 = A_1x(x+1)-(x+1)-2x^2.$$

Taking $x=1$ we obtain

$$A_1 = 2.$$

Hence,

$$\int \frac{x-1}{x^3+x^2}\,dx = 2\int \frac{dx}{x} - \int \frac{dx}{x^2} - 2\int \frac{dx}{x+1} = 2\ln|x| + \frac{1}{x} - 2\ln|x+1| + C.$$

(vii) $\displaystyle\int \frac{1}{(x+4)(x-1)}\,dx.$

**Solution:** Special Case, Partial Fractions: Since

$$\frac{1}{(x+4)(x-1)} = \frac{A_1}{x+4} + \frac{A_2}{x-1} = \frac{A_1(x-1)+A_2(x+4)}{(x+4)(x-1)}$$

we have that

$$1 = A_1(x-1)+A_2(x+4).$$

Taking $x=-4$ we get

$$A_1 = -\frac{1}{5}.$$

Taking $x=1$ we get

$$A_2 = \frac{1}{5}.$$

Thus

$$\int \frac{1}{(x+4)(x-1)}\,dx = -\frac{1}{5}\int \frac{dx}{x+4} + \frac{1}{5}\int \frac{dx}{x-1} = -\frac{1}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C.$$

e) Evaluate

(i) $$\int_0^4 x^2 + 3x^{7/2}\,dx.$$

**Solution:**

$$\int_0^4 x^2 + 3x^{7/2}\,dx = \left(\frac{x^3}{3} + \frac{2}{3}x^{9/2}\right)\Bigg|_0^4 = \frac{4^3 + 24^{9/2}}{3}.$$

(ii) $$\int_1^2 \frac{(x+5)^2}{x^4}\,dx.$$

**Solution:**

$$\int_1^2 \frac{(x+5)^2}{x^4}\,dx = \int_1^2 \frac{x^2+10x+25}{x^4}\,dx = \int_1^2 \frac{dx}{x^2} + 10\int_1^2 \frac{dx}{x^3} + 25\int_1^2 \frac{dx}{x^4} = \left(-\frac{1}{x} - \frac{10}{2x^2} - \frac{25}{3x^3}\right)\Bigg|_1^2 = 277/24.$$

**[3]** 11. Find $f(x)$ if $f'(x) = 8x + \sin(x)$ and $f(0) = \pi$.

**[3]** 12. Estimate the area under the graph of $f(x) = \sqrt{x}$ from 0 to 4 using four approximating rectangles and right endpoints.

**[3]** 13. If $f(x)$ and $g(x)$ are continuous functions such that

$$\int_2^0 f(x)\,dx = \pi, \qquad \int_0^2 g(x)\,dx = 6\pi,$$

find the value of

$$\int_0^2 (3g(x) + f(x))\,dx.$$

**[3]** 14. Find $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \sqrt{1+t^3}\,dt.$

15. Find the following indefinite integrals using any appropriate integration technique.

**[3]** a) $\displaystyle\int \frac{4x^2 + x\sqrt{x}}{x^3}\, dx.$

**[4]** b) $\displaystyle\int xe^{x^2+1}\, dx.$

**[4]** c) $\int x^2 \ln x \, dx.$

**[6]** d) $\displaystyle\int \frac{9}{x(x^2+9)}\,dx.$

**[6]** e) $\displaystyle\int \frac{x^3}{\sqrt{16 - x^2}}\, dx.$

[**4**] 16. Evaluate $\displaystyle\int_0^1 \frac{e^x + 1}{e^x + x}\,dx$.