CONCORDIA UNIVERSITY DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP 232 Mathematics for Computer Science Winter 2017

Assignment 4

Due date: April 14, 2017, 11:59 PM.

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1. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. Use strong mathematical induction to show that $P(n)$ is true for $n \geq 18$.

2. Give a recursive definition of each of these sets of ordered pairs of positive integers.

   (a) $S = \{(a, b) : a \in Z^+, b \in Z^+, \text{ and } a + b \text{ is odd}\}$

   (b) $S = \{(a, b) \ : \ a \in Z^+, b \in Z^+, \text{ and } a \,|\, b$

   (c) $S = \{(a, b) \ : \ a \in Z^+, b \in Z^+, \text{ and } 3 \,|\, (a + b)\}$

3. Prove that $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n + 1)! - 1$ for all $n > 0$.

4. Use mathematical induction to prove that for all $n \in N$ we have $21 \mid (4^{n+1} + 5^{2n-1})$.

5. The Fibonacci numbers are defined as: $f_1 = 1$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2} , \qquad \text{for } n \geq 3 .$$

   Give a proof by induction to show that $3 \mid f_{4n}$, for all $n \geq 1$.

6. Give a proof by induction to show that the Fibonacci numbers satisfy

$$f_{n-1} \, f_{n+1} - f_n^2 = (-1)^n , \quad \text{for all } n \in N, \text{ with } n \geq 2 .$$

7. For each of the following relations on the set of all real numbers, determine whether it is reflexive, symmetric, antisymmetric, transitive. Here xRy if and only if:

   (a) $x + 2y = 0$ (b) $x = 2y$
   (c) $x - y$ is a rational number (d) $xy = 0$
   (e) $xy \geq 0$ (f) $x = 1$ or $y = 1$
   (g) x is a multiple of y (h) $xy = 1$

8. Determine the matrix which represents the transitive closures of the following relations on the set $\{a, b, c, d, e\}$:

   (a) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$

   (b) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$

9. List all possible relations on the set $\{0, 1\}$ and determine which of these relations are

   (a) reflexive (b) symmetric (c) antisymmetric (d) transitive

10. Give the equivalence classes of the relation

$$aRb \quad \text{if and only if} \quad a^4 \equiv b^4 (\bmod\ 30)\ ,$$

on the set $\{1, 2, 3, ..., 15\}$.

11. Which of these collections of subsets are partitions of the set of integers?

    (a) the set of even integers and the set of odd integers

    (b) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

    (c) the set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

    (d) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6.

12. Which of the following are partial orders?

    (a) $(, =)$ (b) $(, <)$ (c) $(, \leq)$ (d) $(, \neq)$

COMP 232 Assignment 4 Solution

Question 1

Base case:
$n = 18$, $18 = 7 + 7 + 4$;
$n = 19$, $19 = 7 + 4 + 4 + 4$;
$n = 20$, $20 = 4 + 4 + 4 + 4 + 4$;
$n = 21$, $21 = 7 + 7 + 7$.
Inductive hypothesis: assume when $18 \leq n \leq k$, n is a sum of 4's and 7's.
Inductive step: we need to show when $n = k + 1$, the statement is true.
From IH, we know $n = k - 3$ is a sum of 4's and 7's. Thus, $n = k - 3 + 4 = k + 1$ is also a sum of 4's and 7's.
Thus, the statement is true for all $n \geq 18$.

Question 2

1. $S = \{(a, b) \;:\; a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a + b \text{ is odd}\}$ Solution:
    - Base case 1: $(1, 2) \in S$.
    - Base case 2: $(2, 1) \in S$.
    - Recursive case 1:if $(i, j) \in S$ then $(i + 1, j + 1) \in S$.
    - Recursive case 2: if $(i, j) \in S$ then $(i, j + 2) \in S$.
    - Recursive case 3: if $(i, j) \in S$ then $(i + 2, j) \in S$.
2. $S = \{(a, b) \;:\; a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } a|b\}$ Solution:
    - Base case: $(n, n) \in S$ for all $n \in \mathbb{Z}^+$.
    - Recursive case: if $(i, j) \in S$ then $(i, j \cdot k) \in S$, for all $k \in \mathbb{Z}^+$.
3. $S = \{(a, b) \;:\; a \in \mathbb{Z}^+, b \in \mathbb{Z}^+, \text{ and } 3|(a + b)\}$ Solution:
    - Base case 1: $(1, 2) \in S$.
    - Base case 2: $(2, 1) \in S$.
    - Recursive case 1: if $(i, j) \in S$ then $(i + 3, j) \in S$.
    - Recursive case 2: if $(i, j) \in S$ then $(i, j + 3) \in S$.

- Recursive case 3: if $(i, j) \in S$ then $(i + 1, j + 2) \in S$.
- Recursive case 4: if $(i, j) \in S$ then $(i + 2, j + 1) \in S$.

Question 3

Base case: $n = 1$, $LHS = 1 \cdot 1! = 1$, $RHS = (1 + 1)! - 1 = 2 - 1 = 1$. Thus, $LHS = RHS$.
Inductive hypothesis: Assume when $n = k$, the statement $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$ is true.
Inductive step: We need to show $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$

$$\begin{aligned}
LHS &= (k + 1)! - 1 + (k + 1) \cdot (k + 1)! \\
&= (k + 1)!(1 + k + 1) - 1 \\
&= (k + 2)! - 1 \\
&= RHS
\end{aligned}$$

Thus, the statement is true for all $n \geq 1$.

Question 4

Base case: when $n = 1$, $4^{1+1} + 5^{2-1} = 21 \equiv 0 \mod 21$.
Inductive hypothesis: assume when $n = k$, $4^{k+1} + 5^{2k-1} \equiv 0 \mod 21$.
Inductive step:
By inductive hypothesis:

$$\begin{aligned}
4^{k+1} + 5^{2k-1} &\equiv 0 \mod 21 \\
4^{k+1} + 5^{2k-1} &= 21a \\
5^{2k-1} &= 21a - 4^{k+1}
\end{aligned}$$

The equivalence we want to show:

$$\begin{aligned}
&4^{k+1+1} + 5^{2(k+1)-1} \\
=&4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1} \\
=&4 \cdot 4^{k+1} + 25(21a - 4^{k+1}) \text{ by substitution} \\
=&21 \cdot 4^{k+1} + 25 \cdot 21a \\
=&21(4^{k+1} + 25a) \\
\equiv&0 \mod 21
\end{aligned}$$

Question 5

Base case:
when $n = 1$, $f_{4n} = f_4 = 3 \equiv 0 \mod 3$.

Inductive hypothesis: assume $n = k$, $f_{4n} \equiv 0 \mod 3$.
Inductive step: when $n = k + 1$

$$\begin{aligned} f_{4k+4} &= f_{4k+3} + f_{4k+2} \\ &= f_{4k+2} + 2f_{4k+1} + f_{4k} \\ &= f_{4k+1} + 2f_{4k-1} + 4f_{4k} \\ &= 5f_{4k} + 3f_{4k-1} \end{aligned}$$

From inductive hypothesis, we know $5f_{4k} \equiv 0 \mod 3$. And $3f_{4k-1} \equiv 0 \mod 3$.
Thus, $f_{4k+4} \equiv 0 \mod 3$.

Question 6

Base case: when $n = 2$, $LHS = f_{2-1}f_{2+1} - f_2^2 = 1 \times 2 - 1 = 1 = 1^2 = RHS$.
Inductive hypothesis: assume when $n = k$, $f_{k-1}f_{k+1} - f_k^2 = (-1)^k$.
Inductive step: when $n = k + 1$,

$$\begin{aligned} f_k f_{k+2} - f_{k+1}^2 &= f_k(f_k + f_{k+1}) - f_{k+1}^2 \\ &= f_k^2 + f_k f_{k+1} - f_{k+1}^2 \\ &= f_k^2 - f_{k+1}(f_{k+1} - f_k) \\ &= f_k^2 - f_{k+1}(f_k + f_{k-1} - f_k) \\ &= f_k^2 - f_{k+1}f_{k-1} \\ &= -(-1)^k \\ &= (-1)^{k+1} \end{aligned}$$

Thus, the statement is true for all $n \geq 2$.

Question 7

	Reflexive	Symmetric	Antisymmetric	Transitive
a	No	No	Yes	No
b	No	No	Yes	No
c	Yes	Yes	No	Yes
d	No	Yes	No	No
e	Yes	Yes	No	No
f	No	Yes	No	No
g	Yes	No	Yes	Yes
h	No	Yes	No	No

Question 8

a. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Question 9

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Question 10

$$\begin{aligned} [1] &= \{1, 7, 11, 13\} \\ [2] &= \{2, 4, 8, 14\} \\ [3] &= \{3, 9\} \\ [5] &= \{5\} \\ [6] &= \{6, 12\} \\ [10] &= \{10\} \\ [15] &= \{15\} \end{aligned}$$

Question 11

a) Yes

b) Yes

c) Yes

d) No

Question 12

a) Yes

b) No

c) Yes

d) No