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Homework for 1/27 Due 2/5

1. [§8-13] In Example D of Section 8.4, the pdf of the population distribution is

$$f(x|\alpha) = \begin{cases} \dfrac{1+\alpha x}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \qquad -1 \le \alpha \le 1,$$

and the method of moments estimate was found to be $\hat{\alpha} = 3\overline{X}$ (where $\overline{X}$ is the sample mean of the random sample $X_1, \dots, X_n$). In this problem, you will consider the sampling distribution of $\hat{\alpha}$.

   (a) Show that the estimate $\hat{\alpha}$ is unbiased.

   (b) Find $\mathrm{Var}[\hat{\alpha}]$. [*Hint:* What is $\mathrm{Var}[\overline{X}]$?]

   (b) Use the central limit theorem to deduce a normal approximation to the sampling distribution of $\hat{\alpha}$. According to this approximation, if $n = 25$ and $\alpha = 0$, what is the $\mathbb{P}(|\hat{\alpha}| > .5)$?

(a) We notice that

$$\begin{aligned}\mathbb{E}\left[\overline{X}\right] = \mathbb{E}[X_1] &= \int_{-1}^{1} x \cdot \frac{1+\alpha x}{2}\, dx = \int_{-1}^{1} \left(\frac{x}{2} + \frac{\alpha x^2}{2}\right) \\ &= \left(\frac{x^2}{4} + \frac{\alpha x^3}{6}\right)\Big|_{-1}^{1} = \frac{\alpha}{3}.\end{aligned}$$

Therefore,

$$\mathbb{E}[\hat{\alpha}] = \mathbb{E}\left[3\overline{X}\right] = 3\mathbb{E}\left[\overline{X}\right] = 3 \cdot \frac{\alpha}{3} = \alpha.$$

(b) First, we have

$$\begin{aligned}\mathbb{E}[X_1^2] &= \int_{-1}^{1} x^2 \cdot \frac{1+\alpha x}{2}\, dx = \int_{-1}^{1} \left(\frac{x^2}{2} + \frac{\alpha x^3}{2}\right) \\ &= \left(\frac{x^3}{6} + \frac{\alpha x^4}{8}\right)\Big|_{-1}^{1} = \frac{1}{3},\end{aligned}$$

and

$$\mathrm{Var}[X_1] = \mathbb{E}[X_1^2] - \mathbb{E}[X_1]^2 = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \frac{3-\alpha^2}{9}.$$

Thus,

$$\begin{aligned}\mathrm{Var}\left[\overline{X}\right] &= \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left[X_i\right] \\ &= \frac{1}{n}\mathrm{Var}[X_1] = \frac{3-\alpha^2}{9n}.\end{aligned}$$

Therefore, we have

$$\mathrm{Var}[\hat{\alpha}] = \mathrm{Var}\left[3\overline{X}\right] = 9\mathrm{Var}\left[\overline{X}\right] = \frac{3-\alpha^2}{n}.$$

(c) According to the central limit theorem, we have $\overline{X} \sim N\left(\frac{\alpha}{3}, \frac{3-\alpha^2}{9n}\right)$, approximately. Therefore, $\hat{\alpha} = 3\overline{X}$ implies that

$$\hat{\alpha} \sim N\left(\alpha, \frac{3-\alpha^2}{n}\right), \qquad \text{approximately.}$$

In the case $\alpha = 0$ and $n = 25$, we have $\hat{\alpha} \sim N(0, 0.12)$, approximately. Thus

$$\begin{aligned}
\mathbb{P}(|\hat{\alpha}| > .5) &= \mathbb{P}(\hat{\alpha} > .5) + \mathbb{P}(\hat{\alpha} < -.5) \\
&= \mathbb{P}\left(\frac{\hat{\alpha}-0}{\sqrt{0.12}} > \frac{0.5-0}{\sqrt{0.12}}\right) + \mathbb{P}\left(\frac{\hat{\alpha}-0}{\sqrt{0.12}} < \frac{-0.5-0}{\sqrt{0.12}}\right) \\
&\approx \mathbb{P}(Z > 1.44) + \mathbb{P}(Z < -1.44) \\
&= 0.0749 + 0.0749 = 0.1498.
\end{aligned}$$

2. [§8-53] Let $X_1, \ldots, X_n$ be i.i.d. uniform on $[0, \theta]$.

(a) Find the method of moments estimate of θ, and the mean, variance, bias, and MSE of the MME.

(b) The mle of θ is $\hat{\theta} = \max_{1 \le i \le n} X_i$. The pdf of $\max_{1 \le i \le n} X_i$ (*How do we find this?*) is

$$f(x|\theta) = \begin{cases} \dfrac{nx^{n-1}}{\theta^n} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}.$$

Calculate the mean and variance of the mle. Compare the variance, the bias, and the mean squared error to those of the method of moments estimate.

(c) Find a modification of the mle that renders it unbiased.

(a) Since

$$\mu_1 = \mathbb{E}[X_1] = \int_0^\theta x \frac{1}{\theta}\, dx = \left.\frac{x^2}{2\theta}\right|_0^\theta = \frac{\theta}{2},$$

we have

$$\theta = 2\mu_1,$$

and the MME for θ is

$$\tilde{\theta} = 2\overline{X},$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Furthermore, we have

$$\mu_2 = \mathbb{E}[X_1^2] = \int_0^\theta x^2 \frac{1}{\theta}\, dx = \left.\frac{x^3}{3\theta}\right|_0^\theta = \frac{\theta^2}{3},$$

and

$$\mathrm{Var}[X_1] = \mu_2 - \mu_1^2 = \frac{\theta^2}{3} - \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{12}.$$

Thus

$$\mathbb{E}\left[\overline{X}\right] = \mathbb{E}[X_1] = \frac{\theta}{2}, \quad \text{and} \quad \mathrm{Var}\left[\overline{X}\right] = \frac{1}{n}\mathrm{Var}[X_1] = \frac{\theta^2}{12n}.$$

It follows that

$$\mathbb{E}[\tilde{\theta}] = \mathbb{E}\left[2\overline{X}\right] = 2\mathbb{E}\left[\overline{X}\right] = \theta,$$

and

$$\mathrm{Var}[\tilde{\theta}] = \mathrm{Var}\left[2\overline{X}\right] = 4\mathrm{Var}\left[\overline{X}\right] = \frac{\theta^2}{3n}.$$

In particular, the MME $\tilde{\theta}$ is unbiased. The bias and MSE of $\tilde{\theta}$ are

$$b(\tilde{\theta}) = 0,$$

and

$$\mathrm{MSE}(\tilde{\theta}) = \mathrm{Var}[\tilde{\theta}] + b(\tilde{\theta})^2 = \mathrm{Var}[\tilde{\theta}] = \frac{\theta^2}{3n}.$$

(b) The mean of $\hat{\theta}$ is

$$\mathbb{E}[\hat{\theta}] = \int_0^\theta x \cdot \frac{nx^{n-1}}{\theta^n}\, dx = \left.\left(\frac{nx^{n+1}}{(n+1)\theta^n}\right)\right|_0^\theta = \frac{n}{n+1}\theta.$$

Since

$$\mathbb{E}[\hat{\theta}^2] = \int_0^\theta x^2 \cdot \frac{nx^{n-1}}{\theta^n}\, dx = \left.\left(\frac{nx^{n+2}}{(n+2)\theta^n}\right)\right|_0^\theta = \frac{n}{n+2}\theta^2,$$

the variance of $\hat{\theta}$ is

$$\mathrm{Var}[\hat{\theta}] = \mathbb{E}[\hat{\theta}^2] - \mathbb{E}[\hat{\theta}]^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}.$$

The bias of $\hat{\theta}$ is

$$b(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta = -\frac{\theta}{n+1},$$

and the MSE of $\hat{\theta}$ is

$$\begin{aligned}\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}[\hat{\theta}] + b(\hat{\theta})^2 &= \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2}{(n+1)^2} \\ &= \frac{2\theta^2}{(n+1)(n+2)}.\end{aligned}$$

By comparison, although the MLE $\hat{\theta}$ is biased while the MME $\tilde{\theta}$ is unbiased, we see that $\mathrm{MSE}(\hat{\theta}) < \mathrm{MSE}(\tilde{\theta})$ when n is large. In fact, $\mathrm{MSE}(\hat{\theta})$ decreases much faster than $\mathrm{MSE}(\tilde{\theta})$.

(c) Let

$$\bar{\theta} = \frac{n+1}{n}\hat{\theta} = \frac{n+1}{n}\max_{1\le i\le n} X_i,$$

it follows that

$$\mathbb{E}\left[\bar{\theta}\right] = \mathbb{E}\left[\frac{n+1}{n}\hat{\theta}\right] = \frac{n+1}{n}\mathbb{E}\left[\hat{\theta}\right] = \theta.$$

Thus $\bar{\theta}$ is unbiased.

3. [§8-57] This problem is concerned with the estimation of the variance of a normal distribution with unknown mean from a sample $X_1, \ldots, X_n$ of i.i.d. normal random variables $N(\mu, \sigma^2)$. In answering the following questions, use the fact that (from Theorem B of Section 6.3)

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

and that the mean and variance of a chi-square random variable with r df are r and $2r$, respectively.

(a) Which of the following estimates is unbiased?

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

(*We discussed this in class. However, we do not assume normality. When the distribution is not normal, the argument is much more complicated, as seen in class. A technical detail is provided at the end of this homework.*)

(b) Which of the estimates given in part (a) has the smaller MSE?

(c) For what value of ρ does $\rho\sum_{i=1}^{n}(X_i - \overline{X})^2$ have the minimal MSE (as an estimate for σ^2)?

(a) Since

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1},$$

and

$$\mathbb{E}[U] = r \quad \text{and} \quad \mathrm{Var}[U] = 2r,$$

where $U \sim \chi^2_r$, we have

$$\mathbb{E}\left[\frac{(n-1)s^2}{\sigma^2}\right] = n-1 \quad \text{and} \quad \mathrm{Var}\left[\frac{(n-1)s^2}{\sigma^2}\right] = 2(n-1).$$

Therefore,

$$\mathbb{E}\left[s^2\right] = (n-1)\cdot\frac{\sigma^2}{(n-1)} = \sigma^2,$$

and

$$\mathrm{Var}\left[s^2\right] = 2(n-1)\cdot\left(\frac{\sigma^2}{(n-1)}\right)^2 = \frac{2\sigma^4}{n-1}.$$

Furthermore, since $\hat{\sigma}^2 = \dfrac{n-1}{n}s^2$, we have

$$\mathbb{E}[\hat{\sigma}^2] = \frac{n-1}{n}\mathbb{E}[s^2] = \frac{n-1}{n}\sigma^2,$$

and

$$\mathrm{Var}[\hat{\sigma}^2] = \left(\frac{n-1}{n}\right)^2 \mathrm{Var}[s^2] = \frac{2(n-1)}{n^2}\sigma^4.$$

Thus s^2 is an unbiased estimate for σ^2 while $\hat{\sigma}^2$ is not.

(b) The bias of the two estimates are

$$b(s^2) = 0 \quad \text{and} \quad b(\hat{\sigma}^2) = \mathbb{E}[\hat{\sigma}^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2,$$

respectively. Thus the MSE of the two estimates are

$$\mathrm{MSE}(s^2) = \mathrm{Var}[s^2] + b(s^2)^2 = \frac{2\sigma^4}{n-1} + 0^2 = \frac{2\sigma^4}{n-1},$$

and

$$\begin{aligned}\mathrm{MSE}(\hat{\sigma}^2) = \mathrm{Var}[\hat{\sigma}^2] + b(\hat{\sigma}^2)^2 &= \frac{2(n-1)}{n^2}\sigma^4 + \left(-\frac{1}{n}\sigma^2\right)^2 \\ &= \frac{2n-1}{n^2}\sigma^4,\end{aligned}$$

respectively. Since

$$3n > 1 \Rightarrow (2n-1)(n-1) < 2n^2 \Rightarrow \frac{2n-1}{n^2} < \frac{2}{n-1},$$

we have $\mathrm{MSE}(\hat{\sigma}^2) < \mathrm{MSE}(s^2)$.

(c) Let $Y := \rho\sum_{i=1}^{n}(X_i - \overline{X})^2$. Then $Y = \rho(n-1)s^2$. By a similar argument as in (a), we have

$$\mathbb{E}[Y] = \rho(n-1)\mathbb{E}[s^2] = \rho(n-1)\sigma^2,$$

and

$$\mathrm{Var}[Y] = (\rho(n-1))^2\,\mathrm{Var}[s^2] = \rho^2(n-1)^2\frac{2\sigma^4}{n-1} = 2\rho^2(n-1)\sigma^4.$$

Thus

$$\begin{aligned}\mathrm{MSE}(Y) = \mathrm{Var}[Y] + b(Y)^2 &= 2\rho^2(n-1)\sigma^4 + \left(\rho(n-1)\sigma^2 - \sigma^2\right)^2 \\ &= \sigma^4[2\rho^2(n-1) + (\rho n - \rho - 1)^2] \equiv f(\rho).\end{aligned}$$

Since

$$\begin{aligned}f'(\rho) &= \sigma^4[4\rho(n-1) + 2(\rho n - \rho - 1)(n-1)] \\ &= \sigma^4[4\rho + 2(\rho n - \rho - 1)](n-1) \\ &= 2(n-1)\sigma^4(\rho n + \rho - 1),\end{aligned}$$

and

$$f''(\rho) = 2(n-1)(n+1)\sigma^4 > 0,$$

we see that $f(\rho)$ achieve its minimum at $\rho = \dfrac{1}{n+1}$. ($f\left((n+1)^{-1}\right) = 0$.)

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Homework for 1/29 Due 2/5

1. [§8-7] Suppose that X follows a geometric distribution,

$$\mathbb{P}(X = k) = p(1-p)^{k-1}$$

and assume $X_1, \ldots, X_n$ is an i.i.d. sample of size n. Find the asymptotic variance of the mle. (*The moments of geometric distribution can be found in P117.*)

We have

$$\begin{aligned}
&\log f(X|p) = \log p + (X-1)\log(1-p), \\
&\frac{\partial}{\partial p}\log f(X|p) = \frac{1}{p} - \frac{X-1}{1-p}, \quad \text{and} \\
&\frac{\partial^2}{\partial p^2}\log f(X|p) = -\frac{1}{p^2} - \frac{X-1}{(1-p)^2}.
\end{aligned}$$

Therefore, the Fisher information is

$$\begin{aligned}
I(p) = -\mathbb{E}\left[\frac{\partial^2}{\partial p^2}\log f(X|p)\right] &= -\left(-\frac{1}{p^2} - \frac{1}{(1-p)^2}\left(\mathbb{E}[X]-1\right)\right) \\
&= \frac{1}{p^2} + \frac{1}{(1-p)^2}\left(\frac{1}{p} - 1\right) = \frac{1}{(1-p)p^2},
\end{aligned}$$

and the asymptotic variance of the mle is

$$\frac{1}{nI(p)} = \frac{(1-p)p^2}{n}.$$

2. [§8-16] Consider an i.i.d. sample of random variables with density function

$$f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right), \quad -\infty < x < \infty, \quad \sigma > 0.$$

Find the asymptotic variance of the mle.

We have

$$
\begin{aligned}
&\log f(X|\sigma) = -\log 2 - \log \sigma - \frac{|X|}{\sigma}, \\
&\frac{\partial}{\partial \sigma} \log f(X|\sigma) = -\frac{1}{\sigma} + \frac{|X|}{\sigma^2}, \qquad \frac{\partial^2}{\partial \sigma^2} \log f(X|\sigma) = \frac{1}{\sigma^2} - \frac{2|X|}{\sigma^3}, \qquad \text{and} \\
&\mathbb{E}[|X|] = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx = 2\int_0^{\infty} \frac{x}{2\sigma} \exp\left(-\frac{x}{\sigma}\right) dx \\
&\qquad = \sigma \int_0^{\infty} \frac{x}{\sigma} \exp\left(-\frac{x}{\sigma}\right) d\frac{x}{\sigma} = \sigma \int_0^{\infty} y e^{-y}\, dy \\
&\qquad = \sigma.
\end{aligned}
$$

Therefore, the Fisher information is

$$
I(\sigma) = -\mathbb{E}\left[\frac{\partial^2}{\partial \sigma^2} \log f(X|\sigma)\right] = -\left(\frac{1}{\sigma^2} - \frac{2}{\sigma^3}\mathbb{E}[|X|]\right) = \frac{1}{\sigma^2},
$$

and the asymptotic variance of the mle is

$$
\frac{1}{nI(\sigma)} = \frac{\sigma^2}{n}.
$$

3. [§8-47] The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

$$f(x|x_0, \theta) = \theta x_0^\theta x^{-\theta-1}, \qquad x \geq x_0, \theta > 1.$$

Assume that $x_0 > 0$ is given and that $X_1, X_2, \dots, X_n$ is an i.i.d. sample.

(a) Find the method of moments estimate of θ.

(b) Find the mle of θ.

(c) Find the asymptotic variance of the mle.

(a) Since

$$\begin{aligned}\mu_1 = \mathbb{E}[X_1] &= \int_{x_0}^\infty x \cdot \theta x_0^\theta x^{-\theta-1}\, dx = \theta x_0^\theta \int_{x_0}^\infty x^{-\theta}\, dx \\ &= \theta x_0^\theta \frac{x^{1-\theta}}{1-\theta}\bigg|_{x_0}^\infty = \frac{\theta x_0}{\theta - 1},\end{aligned}$$

we have

$$\theta = \frac{\mu_1}{\mu_1 - x_0},$$

and the MME is

$$\tilde{\theta} = \frac{\overline{X}_1}{\overline{X} - x_0},$$

where $\overline{X} = \sum_{i=1}^n X_i$.

(b) The log likelihood function is

$$\begin{aligned}\log f(\theta) &= \sum_{i=1}^n (\log\theta + \theta\log x_0 - (\theta+1)\log X_i) \\ &= n\log\theta + n\theta\log x_0 - (\theta+1)\sum_{i=1}^n \log X_i.\end{aligned}$$

Thus

$$\begin{aligned} l'(\theta) &= \frac{n}{\theta} + n\log x_0 - \sum_{i=1}^n \log X_i, \qquad \text{and} \\ l''(\theta) &= -\frac{n}{\theta^2} < 0.\end{aligned}$$

Since

$$l'\left(\frac{1}{\frac{1}{n}\sum_{i=1}^n \log X_i - \log x_0}\right) = 0,$$

the mle of θ is
$$\hat{\theta} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \log X_i - \log x_0}.$$

(c) We have

$$\log f(X|\theta) = \log\theta + \theta \log x_0 - (\theta+1)\log X,$$
$$\frac{\partial}{\partial\theta}\log f(X|\theta) = \frac{1}{\theta} + \log x_0 - \log X \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(X|\theta) = -\frac{1}{\theta^2}.$$

Therefore, the Fisher information is

$$I(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right] = -\left(-\frac{1}{\theta^2}\right) = \frac{1}{\theta^2},$$

and the asymptotic variance of the mle is

$$\frac{1}{nI(\theta)} = \frac{\theta^2}{n}.$$