MAT 1332, Summer 2017, Assignment 4
Due FRI, JUNE 23 in the math department dropboxes by 3:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

Instructor Catalin Rada

Student Name ______________________________ Student Number ______________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________________

QUESTION 1. For the system of linear equations

$$\begin{array}{ccccccc} x & + & 3y & + & 9z & = & 3 \\ 2x & + & 7y & + & 23z & = & 2 \\ x & + & ay & + & a^2z & = & a \end{array}$$

a) determine the values of a for which the system has

(i) no solution,
(ii) infinitely many solutions,
(iii) a unique solution.

The augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & a & a^2 & a \end{array}\right]$$

We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - R_1$, $R_3 \rightsquigarrow R_3 - (a-3)R_2$, and obtain

$$A \sim \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & a-3 & a^2-9 & a-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & a^2-9-5a+15 & 5(a-3) \end{array}\right].$$

Since $a^2 - 9 - 5a + 15 = (a-3)(a-2)$ we get:

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & (a-3)(a-2) & 5(a-3) \end{array}\right]$$

- If $a = 2$, then the last row of the matrix is is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -5 \end{array}\right]$. Hence the system is inconsistent.
- If $a = 3$ then
$$M = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -6 & 10 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
Hence the system has infinitely many solutions.
- If $a \notin \{3, 2\}$, then $(a-3)(a-2) \neq 0$ and so the system is uniquely solvable

The answer to question (a) is therefore:

[1] (i) The system in inconsistent if $a = 2$.

[1] (ii) The system has infinitely many solutions if $a = 3$.

[1] (iii) The system is uniquely solvable if $a \notin \{2, 3\}$.

b) In case (ii) above describe all solutions.

The RREF of the matrix is
$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 15 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding linear system is
$$\begin{array}{ccccccc} x & & & - & 6z & = & 15 \\ & & y & + & 5z & = & -4 \end{array}$$

Thus z is a free variable. Putting $z = t$, the general solution is
$$(15 + 6t, -4 - 5t, t) \quad (t \text{ a free parameter})$$

c) If $a = 1$ find the inverse of the matrix
$$A = \begin{bmatrix} 1 & 3 & 9 \\ 2 & 7 & 23 \\ 1 & a & a^2 \end{bmatrix}$$

[2] We set up the 3 by 6 matrix for the inverse.
$$\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 2 & 7 & 23 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

(0.5 points for the correct approach) We use Gauss–Jordan elimination on this matrix
$$R_2 \rightsquigarrow R_2 - 2R_1, R_3 \rightsquigarrow R_3 - R_1 \left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & -2 & -8 & -1 & 0 & 1 \end{array}\right].$$

Next: $R_3 \rightsquigarrow R_3 + 2R_2$ and then $R_3 \rightsquigarrow 0.5R_3$: $\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$. The next step is $R_1 \rightsquigarrow R_1 - 3R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 7 & -3 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$ and finally $R_1 \rightsquigarrow R_1 + 6R_3, R_2 \rightsquigarrow R_2 - 5R_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 3 & 3 \\ 0 & 1 & 0 & -2 & -4 & -2.5 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$. **(1 point for row operations, no points if the left hand side is not reduced to RREF.)** So the inverse is

$$\begin{bmatrix} -8 & 3 & 3 \\ -10.5 & -4 & -2.5 \\ -2.5 & 1 & 0.5 \end{bmatrix}.$$

(0.5 for the correct answer or consistent with previous work.)

QUESTION 2. Given the following matrices and vectors

$$A = \begin{bmatrix} 1 & -7 & -9 \\ 6 & 0 & 8 \\ 4 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 6 & 4 \\ 0 & 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 7/4 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1/2 \\ 3/4 \end{bmatrix}.$$

Compute the following if possible. If not possible, explain in one sentence why.

a) $3\mathbf{v} + 2B^T\mathbf{w}$.

$3\mathbf{v} + 2B^T\mathbf{w}$ is not defined since $B^T\mathbf{w}$ is 2×1 and $\mathbf{v}$ is 3×1.

b) $\mathbf{w}\mathbf{v}^T$

[1]

$$\mathbf{w}\mathbf{v}^T = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 7/8 & 0 \\ 3/4 & 21/6 & 0 \end{bmatrix}.$$

c) $\mathbf{v}^T\mathbf{w}$

$\mathbf{v}^T\mathbf{w} = 1 \times 0 + 7/4 \times 1/2 + 0 \times 3/4 = 7/8$.

d) $A^TB + 2\mathbf{v}^T\mathbf{w}$

$A^TB + 2\mathbf{v}^T\mathbf{w}$ is not defined since A^TB is a 3×2 matrix and $\mathbf{v}^T\mathbf{w}$ is 1×1.

e) AB

[1]

$$AB = \begin{bmatrix} -39 & 44 \\ 0 & 8 \\ 36 & -12 \end{bmatrix}.$$

f) $B\mathbf{u}$

$B\mathbf{u}$ is not defined since B has 2 columns and $\mathbf{u}$ has 3 rows.

g) BA

[1] BA is not defined since B has 2 columns and A has 3 rows.

h) A^2

$$A^2 = \begin{bmatrix} -77 & -43 & -65 \\ 38 & -10 & -54 \\ 28 & -28 & -4 \end{bmatrix}.$$

Question 3. Determine the matrix A such that:

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T.$$

We calculate the both sides of the equation

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = 3(A^T)^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}^T = 3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & 9 \\ -5 & 12 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 21 & 27 \\ -15 & 36 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

Hence

$$3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

Therefore

$$3A = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 6+9 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 6 & \frac{26}{3} \\ \frac{1}{3}(-10) & \frac{1}{3}(47) \\ \frac{8}{3} & \frac{4}{3}(4) \end{bmatrix}.$$

Question 4.

a) Express $z_1 = e^{i\pi/2}$ and $z_2 = e^{i\pi/6}$ in the form $z = a + ib$.

$$z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$

$$z_2 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

b) Express z_1/z_2 in the form $z = a + ib$.

$$\frac{z_1}{z_2} = \frac{e^{i\pi/2}}{e^{i\pi/6}} = e^{i(\pi/2-\pi/6)} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

c) Express $\omega = -12 + 5i$ in the form $\omega = re^{i\theta}$.

We have $r = \sqrt{(-12)^2 + 5^2} = 13$. Since the vector is in the third quadrant, we have

$$\theta = \pi - \arctan\frac{5}{12} = 2.7468.$$

Thus,

$$\omega = 13e^{2.7468i}$$

d) Find $\omega\bar{\omega}$.

[1] We have $\bar{\omega} = 13e^{-2.7468i}$, so

$$\omega\bar{\omega} = 13e^{2.7468i}13e^{-2.7468i} = 169.$$

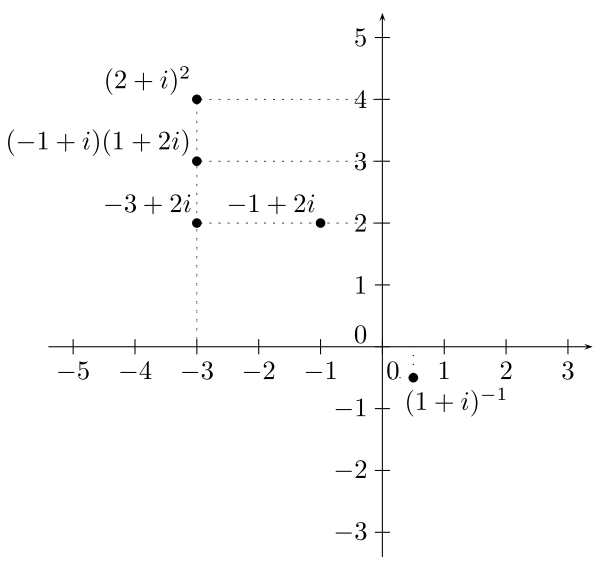
Alternatively, we could just have multiplied out:

$$\omega\bar{\omega} = (-12 + 5i)(-12 - 5i) = 144 - 25i^2 = 169.$$

QUESTION 5. Draw the following complex numbers in the plane:

$$-1 + 2i, \quad -3 + 2i, \quad (-1 + i)(1 + 2i), \quad (2 + i)^2, \quad (1 + i)^{-1}.$$

[6] Solution: $(-1 + i)(1 + 2i) = -3 - \mathrm{i}$ $\quad (2 + i)^2 = 3 + 4\mathrm{i}$, $\quad (1 + i)^{-1} = \frac{1}{2} - \frac{1}{2}\mathrm{i}$



3 points for sketch, 1 point for finding the correct cartesian coordinates of each of $(-1 + i)(1 + 2i)$, $\quad (2 + i)^2$, $\quad (1 + i)^{-1}$**.**