1. Consider the solid in the first octant bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x = 1$, $y = 2$ and the surface $z = 4 - xy$. This solid has a mass density given by the function $\delta(x, y, z) = x + 3y$. Find the total mass of this solid.

$$\text{Mass} = \iiint_R \delta \, dV = \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{4-xy} (x+3y)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^2 (x+3y)z \Big|_0^{4-xy} dy\, dx = \int_0^1 \int_0^2 (x+3y)(4-xy)\, dy\, dx$$

$$= \int_0^1 \int_0^2 \left(4x - x^2y + 12y - 3xy^2\right) dy\, dx = \int_0^1 4xy - \frac{x^2y^2}{2} + 6y^2 - xy^3 \Big|_0^2 dx$$

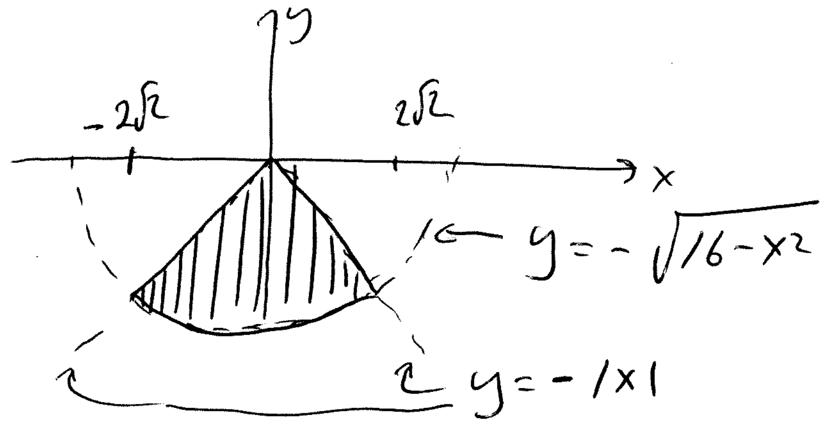
$$= \int_0^1 \left(\cancel{8x} - 2x^2 + 24 - \cancel{8x}\right) dx = 24x - \frac{2x^3}{3}\Big|_0^1 = 24 - \frac{2}{3}$$

$$= 70/3$$

2. Compute the following double integral

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-\sqrt{16-x^2}}^{-|x|} \sqrt{(x^2+y^2)}\, dy\, dx$$

Hint: Sketch the region of integration as a type I region in the x-y plane, and then express this region and the integral in a different coordinate system.



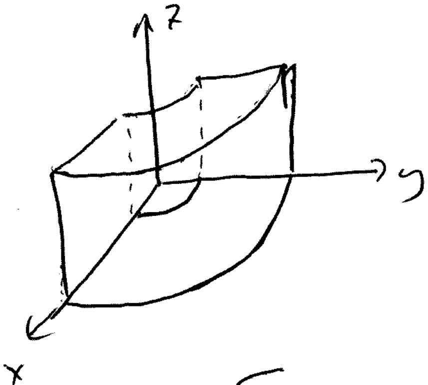
In polar coordinates, the region of integration is described as $0 \le r \le 4$, $5\pi/4 \le \theta \le 7\pi/4$ $\Longrightarrow$

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-\sqrt{16-x^2}}^{-|x|} \sqrt{(x^2+y^2)}\, dy\, dx = \int_{\theta=5\pi/4}^{7\pi/4} \int_{r=0}^{4} \sqrt{r^2}\, \overbrace{r\,dr\,d\theta}^{dA \text{ in polar coords}}$$

$$= \int_{5\pi/4}^{7\pi/4} \int_0^4 r^2\, dr\, d\theta = \int_{5\pi/4}^{7\pi/4} \left. \frac{r^3}{3} \right|_0^4 d\theta = \int_{5\pi/4}^{7\pi/4} \frac{64}{3}\, d\theta = \frac{64}{3} \cdot \frac{\pi}{2}$$

$$= \frac{32\pi}{3}$$

3. Consider the solid in the first octant which is bounded by the planes $z = 0$, $z = 3$, $y = 0$, $x = 0$ and the surfaces $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$. This solid has a mass density given by the function $\delta(x, y, z) = z^3x^2 + z^3y^2$. Set up a triple integral in **cyclindrical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



In cylindrical coordinates, the region of integration is

$1 \le r \le 4$
$0 \le \theta \le \pi/2$
$0 \le z \le 3$

$$\text{So Mass} = \iiint_R \delta \, dV = \int_{z=0}^{3} \int_{\theta=0}^{\pi/2} \int_{r=1}^{4} z^3(r^2) \underbrace{r\,dr\,d\theta\,dz}_{dV}$$

$$= \int_0^3 \int_0^{\pi/2} \int_1^4 r^3 z^3 \, dr \, d\theta \, dz.$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = 3t\,\vec{\mathbf{i}} + 4\sin(2t)\,\vec{\mathbf{j}} + 4\cos(2t)\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/4.$$

$$\vec{r}\,'(t) = 3\vec{i} + 8\cos(2t)\,\vec{j} - 8\sin(2t)\,\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{9 + 64\,(\underbrace{\cos^2(2t) + \sin^2(2t)}_{=1})}$$

$$= \sqrt{9+64} = \sqrt{73}$$

$$\Rightarrow \text{Length} = \int_0^{\pi/4} \|\vec{r}\,'(t)\|\,dt = \int_0^{\pi/4} \sqrt{73}\,dt = \sqrt{73}\,\pi/4.$$

Using cylindrical coordinates

5. Consider the surface which is the portion of the sphere $x^2+y^2+z^2=1$ that lies between the planes $z=-\frac{\sqrt{2}}{2}$ and $z=\frac{\sqrt{2}}{2}$. (See figures on next page).

(a) Give a parametrization of this surface.

(b) Set up an integral which would give the total area of this surface, **but do not evaluate this integral**.

(c) **BONUS [2 marks]** Evaluate the integral in (b). Note that you are eligible to receive bonus marks **only if you have the correct answer in (b).**

a) $x^2+y^2+z^2=1$ intersects $z=\pm\sqrt{2}/2$ when $x^2+y^2+\frac{1}{2}=1$, i.e. $x^2+y^2=1/2$. So, using $x=a\cos\theta$, $y=a\sin\theta$, $z=z$ and

$$a^2+z^2=1 \Rightarrow a=\sqrt{1-z^2}$$

we have $\vec{r}(\theta,z)=\sqrt{1-z^2}\cos\theta\,\vec{i}+\sqrt{1-z^2}\sin\theta\,\vec{j}+z\vec{k}$

$$-\frac{\sqrt{2}}{2}\le z\le\frac{\sqrt{2}}{2},\quad 0\le\theta\le 2\pi$$

b) $\vec{r}_\theta=-\sqrt{1-z^2}\sin\theta\,\vec{i}+\sqrt{1-z^2}\cos\theta\,\vec{j}$

$$\vec{r}_z=\frac{-z}{\sqrt{1-z^2}}\cos\theta\,\vec{i}-\frac{z}{\sqrt{1-z^2}}\sin\theta\,\vec{j}+\vec{k}$$

$$\vec{r}_\theta\times\vec{r}_z=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\sqrt{1-z^2}\sin\theta & \sqrt{1-z^2}\cos\theta & 0\\ -\frac{z}{\sqrt{1-z^2}}\cos\theta & \frac{-z}{\sqrt{1-z^2}}\sin\theta & 1\end{vmatrix}=\vec{i}\left(\sqrt{1-z^2}\cos\theta\right)-\vec{j}\left(-\sqrt{1-z^2}\sin\theta\right)+\vec{k}\left(z(\sin^2\theta+\cos^2\theta)\right)$$

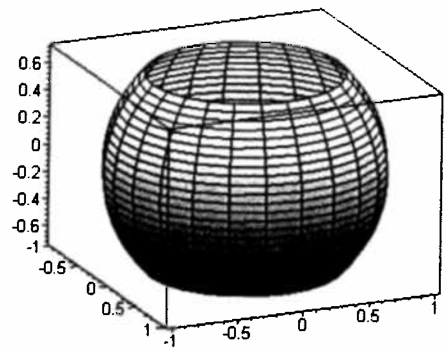
$$=\sqrt{1-z^2}\cos\theta\,\vec{i}+\sqrt{1-z^2}\sin\theta\,\vec{j}+z\vec{k} \Rightarrow \|\vec{r}_\theta\times\vec{r}_z\|=\sqrt{1-z^2+z^2}=\sqrt{1}=1$$

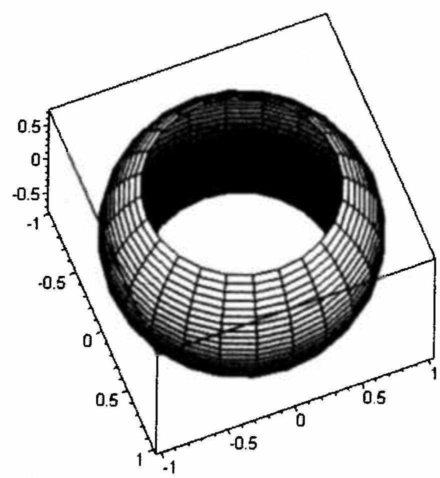
$$\Rightarrow \text{Area}=\int_{\theta=0}^{2\pi}\int_{z=-\sqrt{2}/2}^{\sqrt{2}/2}1\,dz\,d\theta.$$

c) $\text{Area}=2\pi\cdot\left(\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\right)$

$$=2\pi\sqrt{2}.$$

Two different views of the surface described in problem 5.





1. Consider the solid in the first octant bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x = 2$, $y = 1$ and the surface $z = 6 + xy$. This solid has a mass density given by the function $\delta(x, y, z) = 2x + y$. Find the total mass of this solid.

$$\text{Mass} = \iiint_R \delta \, dV = \int_0^2 \int_0^1 \int_0^{6+xy} (2x+y) \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^1 (2x+y) z \Big|_0^{6+xy} dy \, dx = \int_0^2 \int_0^1 (12x + 2x^2 y + 6y + xy^2) \, dy \, dx$$

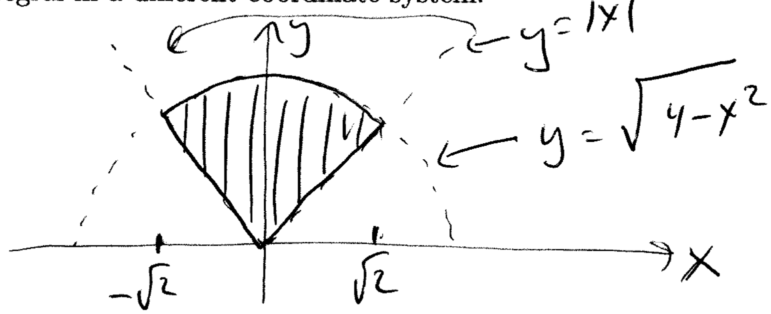
$$= \int_0^2 \left(12xy + x^2 y^2 + 3y^2 + \frac{xy^3}{3} \Big|_0^1 \right) dx = \int_0^2 \left(12x + x^2 + 3 + \frac{x}{3} \right) dx$$

$$= 6x^2 + \frac{x^3}{3} + 3x + \frac{x^2}{6} \Big|_0^2 = 24 + \frac{8}{3} + 6 + \frac{2}{3} = 30 + \frac{10}{3} = \frac{100}{3}$$

2. Compute the following double integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{|x|}^{\sqrt{4-x^2}} \sqrt{(x^2+y^2)}\, dy\, dx$$

Hint: Sketch the region of integration as a type I region in the x-y plane, and then express this region and the integral in a different coordinate system.



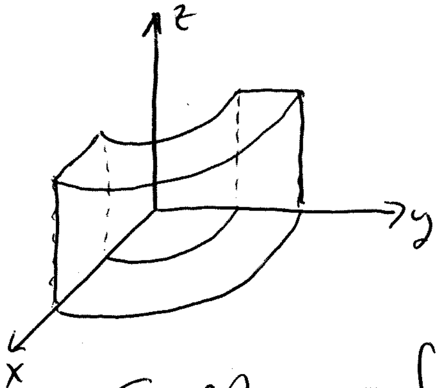
In polar coordinates, the region of integration is described as $0 \le r \le 2$, $\pi/4 \le \theta \le 3\pi/4$ $\Longrightarrow$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{|x|}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_{\theta=\pi/4}^{3\pi/4} \int_{r=0}^{2} \sqrt{r^2}\, \underbrace{r\, dr\, d\theta}_{dA \text{ in polar coords}}$$

$$= \int_{\pi/4}^{3\pi/4} \int_{0}^{2} r^2\, dr\, d\theta = \int_{\pi/4}^{3\pi/4} r^3/3 \Big|_0^2\, d\theta = \int_{\pi/4}^{3\pi/4} 8/3\, d\theta = \frac{8}{3} \cdot \pi/2$$

$$= 4\pi/3$$

3. Consider the solid in the first octant which is bounded by the planes $z = 0$, $z = 2$, $y = 0$, $x = 0$ and the surfaces $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. This solid has a mass density given by the function $\delta(x, y, z) = zx^2 + zy^2$. Set up a triple integral in **cyclindrical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



In cylindrical coordinates, the region of integration is

$1 \le r \le 2$

$0 \le \theta \le \pi/2$

$0 \le z \le 2$

$$\text{So Mass} = \iiint_R \delta \, dV = \int_{z=0}^{2} \int_{\theta=0}^{\pi/2} \int_{r=1}^{2} z r^2 \underbrace{r \, dr \, d\theta \, dz}_{dV}$$

$$= \int_0^2 \int_0^{\pi/2} \int_1^2 r^3 z \, dr \, d\theta \, dz$$

4. Find the total arclength of the curve which is parametrized by the following vector function
$$\vec{r}(t) = 4t\,\vec{\mathbf{i}} + 2\cos(3t)\,\vec{\mathbf{j}} + 2\sin(3t)\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/6.$$

$$\vec{r}\,'(t) = 4\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{16 + 36(\sin^2 3t + \cos^2 3t)}$$

$$= \sqrt{16+36} = \sqrt{52}$$

$$\text{Length} = \int_0^{\pi/6} \|\vec{r}\,'(t)\|\,dt = \int_0^{\pi/6} \sqrt{52}\,dt = \frac{\sqrt{52}\,\pi}{6} = \frac{2\sqrt{13}\,\pi}{6}$$

$$= \frac{\pi\sqrt{13}}{3}$$

Using spherical coordinates

5. Consider the surface which is the portion of the sphere $x^2+y^2+z^2=1$ that lies between the planes $z=-\frac{\sqrt{2}}{2}$ and $z=\frac{\sqrt{2}}{2}$. (See figures on next page).

(a) Give a parametrization of this surface.

(b) Set up an integral which would give the total area of this surface, **but do not evaluate this integral**.

(c) **BONUS** [**2 marks**] Evaluate the integral in (b). Note that you are eligible to receive bonus marks **only if you have the correct answer in (b).**

a) $\vec{r}(\phi,\theta) = \cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}$ $\quad 0\le\theta\le 2\pi$, $\pi/4 \le \theta \le 3\pi/4$

b) $\vec{r}_\phi = \cos\theta\cos\phi\,\vec{i} + \sin\theta\cos\phi\,\vec{j} - \sin\phi\,\vec{k}$

$\vec{r}_\theta = -\sin\theta\sin\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j}$

$$\vec{r}_\phi\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \end{vmatrix} = \sin\phi\left(\cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}\right)$$

$$\|\vec{r}_\theta\times\vec{r}_\phi\| = \sin\phi\,\underbrace{\|\cos\theta\sin\phi\,\vec{i} + \sin\theta\sin\phi\,\vec{j} + \cos\phi\,\vec{k}\|}_{=1} = \sin\phi$$

$$\text{Area} = \int_{\phi=\pi/4}^{3\pi/4}\int_{\theta=0}^{2\pi} \sin\phi\, d\theta\, d\phi$$

c) $$= 2\pi\left(-\cos\phi\,\Big|_{\pi/4}^{3\pi/4}\right) = 2\pi\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}\,\pi$$