

MATH 1104  
FALL 2017  
LECTURE NOTES

©Prepared by Ayşe and Şaban Alaca

These notes replace neither the textbook nor the lectures

Last modified: August 1, 2017

**Part 3**

- Eigenvalues, Eigenvectors and the Characteristic Equation
- Diagonalization
- Complex Numbers
- De Moivre's Theorem
- Roots of a Complex Number
- Complex Eigenvalues and Complex Eigenvectors

### Eigenvalues, Eigenvectors and the Characteristic Equation

**Definition:** Let  $A$  be an  $n \times n$  matrix. A vector  $0 \neq x \in R^n$  is called an **eigenvector** of  $A$  if  $Ax = \lambda x$  for some  $\lambda \in R$ . The scalar  $\lambda$  is called an **eigenvalue** of  $A$ .

**Example:** Let  $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ ,  $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ .

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x.$$

So,  $x$  is an eigenvector of  $A$  corresponding to an eigenvalue  $\lambda = 3$ .

$$Ay = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

So,  $y$  is not an eigenvector of  $A$ .

$\lambda$  is an eigenvalue of  $A \iff Ax = \lambda x$  for some non-zero vector  $x$ .

$\iff (A - \lambda I)x = 0$  for some non-zero vector  $x$ .  $\iff \det(A - \lambda I) = 0$ .

$\det(A - \lambda I) = 0$  is called the **characteristic equation** of  $A$ , the scalars  $\lambda$  satisfying this equation are the eigenvalues of  $A$ .

When expanded,  $\det(A - \lambda I)$  is a polynomial in  $\lambda$ , which has degree  $n$ , and called the **characteristic polynomial** of  $A$ .

The set of all solutions of  $(A - \lambda I)x = 0$  is called the **eigenspace** of  $A$  corresponding to  $\lambda$ , and denoted by  $E_\lambda$ . That is,

$$E_\lambda = \text{Nul}(A - \lambda I)$$

**Remark:** Eigenspace contains the zero vector. But an eigenvector is never the zero vector. An eigenvalue  $\lambda$  might be zero:

$$\lambda = 0 \text{ is an eigenvalue of } A \iff A \text{ is not invertible.}$$

**Example:** Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda = 0 \iff \lambda = 0, 2.$$

Thus,  $A$  is not invertible.

**Example:** Let  $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ .

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 0 & 0 \\ 0 & 0 - \lambda & 0 \\ 1 & 0 & -3 - \lambda \end{vmatrix} = (4 - \lambda)(-\lambda)(-3 - \lambda).$$

$$\det(A - \lambda I) = 0 \iff \lambda_1 = -3, \lambda_2 = 0, \lambda_3 = 4.$$

**Remark:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

**Example:** Let  $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ . Find the eigenvalues and the corresponding eigenvectors of  $A$ . What are the eigenspace(s)?

**Solution:**  $A - \lambda I = \begin{bmatrix} 3 - \lambda & 4 \\ 1 & -\lambda \end{bmatrix}$ . The characteristic polynomial of  $A$  is

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4.$$

The characteristic equation of  $A$  is

$$\lambda^2 - 3\lambda - 4 = 0.$$

The eigenvalues of  $A$ :

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \iff \lambda_1 = 4, \lambda_2 = -1.$$

Eigenvectors for  $\lambda_1 = 4$ :

$$(A - \lambda I)x = 0 \iff \left[ \begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right] \implies x = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad E_4 = \text{Span} \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}.$$

A basis for  $E_4$ :  $\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$ ,  $\dim E_4 = 1$ .

Eigenvectors for  $\lambda_2 = -1$ :

$$A - (-1)I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \iff x = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

A basis for  $E_{-1}$  :  $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ ,  $\dim E_{-1} = 1$ .

**Theorem:** Let  $A$  be an  $n \times n$  matrix.

If  $v_1, v_2, \dots, v_r$  are eigenvectors corresponding to distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_r$ , then  $\{v_1, v_2, \dots, v_r\}$  is a linearly independent set.

**Example:** For  $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$  we know that:

$$\lambda_1 = 4 \implies v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \implies v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

By the above theorem,  $\{v_1, v_2\}$  is a linearly independent set.

### Diagonalization

**Example:** Let  $A$  and  $B$  be two  $n \times n$  matrices such that

$$B = PAP^{-1}$$

for an invertible matrix  $P$ . In this case, we say that  $A$  is similar to  $B$ .

**Exercise:** Show that the similar matrices have the same eigenvalues.

**Solution:** We have  $B - \lambda I = PAP^{-1} - \lambda PP^{-1} = P(A - \lambda I)P^{-1}$ .

$$\begin{aligned} \det(B - \lambda I) &= \det \left( P(A - \lambda I)P^{-1} \right) \\ &= \det P \cdot \det (A - \lambda I) \cdot \det P^{-1} \\ &= \det P \cdot \det (A - \lambda I) \cdot \frac{1}{\det P} \\ &= \det (A - \lambda I). \end{aligned}$$

**Definition:** An  $n \times n$  matrix  $A$  is said to be **diagonalizable** if  $A$  is similar to a diagonal matrix  $D$ , i.e, if  $A = PDP^{-1}$  for some invertible matrix  $P$  and a diagonal matrix  $D$ . We note that

- $A = PDP^{-1} \iff P^{-1}AP = D$ .
- An  $n \times n$  matrix  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.
- An  $n \times n$  matrix  $A$  is diagonalizable if and only if  $R^n$  has a basis consisting of eigenvectors of  $A$ .
- An  $n \times n$  matrix with  $n$  distinct eigenvalues is diagonalizable.
- $A = PDP^{-1} \iff A^n = PD^nP^{-1}$ .

**Example:** Is  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  diagonalizable?

**Solution:** Since  $A$  is upper triangular, the eigenvalues of  $A$  are  $\lambda_1 = \lambda_2 = 0$ .

$$A - \lambda_i I = A \quad (i = 1, 2) \quad \text{and} \quad Ax = 0 \iff x = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, t \in R.$$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is an eigenvector for  $\lambda = 0$ .

$A$  is a  $2 \times 2$  matrix but it does not have 2 linearly independent eigenvectors. Hence,  $A$  is not diagonalizable.

**Example:** Let  $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ . We know that  $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$  is an eigenvector for the eigenvalue  $\lambda_1 = 4$ , and  $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  is an eigenvector for the eigenvalue  $\lambda_2 = -1$ . For

$$P = [v_1 \ v_2] = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

We have  $A = PDP^{-1}$ . (Please verify by showing that  $AP = PD$ .)

$$\begin{aligned} A &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ A^{10} &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}^{10} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{10} & 4^{10} \\ -1 & 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4^{11} + 1 & 4^{11} - 4 \\ 4^{10} - 1 & 4^{11} + 4 \end{bmatrix} = \begin{bmatrix} 838861 & 838860 \\ 209715 & 209716 \end{bmatrix}. \end{aligned}$$

**Example:** Let  $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ .

**Solution:** Since  $A$  is a triangular matrix, eigenvalues of  $A$  are the entries on the main diagonal, i.e,  $\lambda_1 = 1$ ,  $\lambda_2 = -4$ , and  $\lambda_3 = -2$ .

Eigenvectors corresponding to  $\lambda_1 = 1$ :

$$A - \lambda_1 I = A - I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -5 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)X = 0 \iff X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_1 \in R.$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is an eigenvector for  $\lambda_1 = 1$ .

Eigenvectors corresponding to  $\lambda_2 = -4$ :

$$A - \lambda_2 I = A - (-4)I = A + 4I = \begin{bmatrix} 5 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 4I)X = 0 \iff X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/5)x_2 \\ x_2 \\ 0 \end{bmatrix} = (1/5)x_2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

$v_2 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$  is an eigenvector for  $\lambda_2 = -4$ .

Eigenvectors corresponding to  $\lambda_3 = -2$ :

$$A - \lambda_3 I = A - (-2)I = A + 2I = \begin{bmatrix} 3 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2I)X = 0 \iff X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ x_3 \\ x_3 \end{bmatrix} = (1/3)x_3 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

$v_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$  is an eigenvector for  $\lambda_3 = -2$ .

Since  $A$  is a  $3 \times 3$  matrix and it has three linearly independent eigenvectors,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$A$  is diagonalizable.

Let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ and}$$

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}. \text{ Then,}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & -2/15 \\ 0 & 1/5 & -1/5 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

**Exercise:** Let  $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ .

a) Find the eigenvalues and corresponding eigenvectors of  $A$ .

b) Is  $A$  diagonalizable? If yes, write  $A$  as  $A = PDP^{-1}$ , where  $D$  is a diagonal matrix.

**Solution:**

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & 0 & 3 \\ 0 & -1 - \lambda & 0 \\ 1 & 0 & 4 - \lambda \end{vmatrix} \\ &= (-1 - \lambda) \begin{vmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{vmatrix} \\ &= -(1 + \lambda)(\lambda^2 - 6\lambda + 5) \\ &= -(1 + \lambda)(\lambda - 5)(\lambda - 1) = 0 \iff \lambda = -1, 5, 1. \end{aligned}$$

$\lambda_1 = -1$ :

$$A - \lambda I = A + I = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$E_{-1} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \text{a basis for } E_{-1} \text{ is } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \dim E_{-1} = 1.$$

$\lambda_2 = 5$ :

$$A - \lambda I = A - 5I = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 5I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_5 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{a basis for } E_5 \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \dim E_5 = 1$$

$\lambda_3 = 1$ :

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

$$E_1 = \text{Span} \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{a basis for } E_1 \text{ is } \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \dim E_1 = 1.$$

$A$  is a  $3 \times 3$  matrix and the eigenvectors of  $A$

$$\left\{ v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

form a basis for  $\mathbb{R}^3$ .

Thus  $A$  is diagonalizable, and  $A = PDP^{-1}$ , where

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example:** Let  $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ .

- a) Find the eigenvalues and corresponding eigenvectors of  $A$ .  
 b) Diagonalizable  $A$ , if possible.

**Solution:**

$$\det(A - \lambda I) = -(\lambda - 5)^2(\lambda - 1) = 0 \iff \lambda_1 = 5, \lambda_2 = 1.$$

$\lambda_1 = 5$ :

$$A - 5I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 5I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_1 = 5: \quad v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_5 = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \dim E_5 = 2.$$

$\lambda_2 = 1$ :

$$A - I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 \in \mathbb{R}.$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad E_1 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \dim E_1 = 1.$$

Thus  $A = PDP^{-1}$ , where

$$P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Alternatively, we have

$$P_1 = \begin{bmatrix} v_2 & v_1 & v_3 \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \implies A = P_1 D P_1^{-1}.$$

$$P' = \begin{bmatrix} v_3 & v_1 & v_2 \end{bmatrix}, D' = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \implies A = P' D' P'^{-1}.$$

**Example:** Let  $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$ . Is  $A$  diagonalizable?

**Solution:**

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 0 & 3 \\ 1 & -1 - \lambda & 2 \\ -1 & 1 & -2 - \lambda \end{vmatrix} \\ &= (1 - \lambda) \begin{vmatrix} -1 - \lambda & 2 \\ 1 & -2 - \lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 - \lambda \\ -1 & 1 \end{vmatrix} \\ &= (1 - \lambda)(\lambda^2 + 3\lambda) - 3\lambda \\ &= -\lambda^2(\lambda + 2) = 0 \iff \lambda_1 = 0, \lambda_2 = -2. \end{aligned}$$

$\lambda_1 = 0$ :

$$A - \lambda I = A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$Ax = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

$$v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \quad E_0 = \text{Span} \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\lambda_2 = -2$ :

$$A - (-2)I = A + 2I = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}.$$

$$v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad E_{-2} = \text{Span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$A$  is a  $3 \times 3$  matrix but it has only two linearly independent eigenvectors:

$$\left\{ v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\},$$

which is not a basis for  $\mathbb{R}^3$ . Hence,  $A$  is not diagonalizable.

**Example:** Let  $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$ . Diagonalize  $A$ , if possible.

**Solution:**

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 0 & 0 & 2 \\ 0 & 4 - \lambda & 0 & 0 \\ 0 & 0 & 3 - \lambda & 0 \\ 2 & 0 & 0 & 1 - \lambda \end{vmatrix} \\ &= (4 - \lambda) \begin{vmatrix} 1 - \lambda & 0 & 2 \\ 0 & 3 - \lambda & 0 \\ 2 & 0 & 1 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(3 - \lambda) \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(3 - \lambda)(\lambda^2 - 2\lambda - 3) \\ &= (4 - \lambda)(3 - \lambda)(\lambda - 3)(\lambda + 1) = 0 \iff \lambda = -1, 3, 4. \end{aligned}$$

$\lambda_1 = -1$ :

$$A + I = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span}\{v_1\}, \quad \dim E_{-1} = 1.$$

$\lambda_2 = 3$ :

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 3I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_3 = \text{Span}\{v_2, v_3\}, \quad \dim E_3 = 2.$$

$\lambda_3 = 4$ :

$$A - 4I = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 4I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 \in R.$$

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_4 = \text{Span}\{v_4\}, \quad \dim E_4 = 1.$$

$A$  is a  $4 \times 4$  matrix and

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

a basis for  $R^4$ . So  $A$  is diagonalizable.

$$P = [v_1 \ v_2 \ v_3 \ v_4] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$
$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix};$$
$$A = PDP^{-1}$$

### Complex Numbers

A complex number  $z$  is of the form

$$z = a + ib, \text{ where } i^2 = -1, \text{ and } a, b \in R.$$

$a = \operatorname{Re} z$  is the real part of  $z$ , and  $b = \operatorname{Im} z$  is the imaginary part of  $z$ .  
 $z$  is real  $\iff b = 0$ , and  $z$  is purely imaginary  $\iff a = 0$ .

Let  $z = a + ib$  and  $w = c + id$ . Then,

$$z + w = a + c + i(b + d)$$

$$z - w = a - c + i(b - d)$$

$$z \cdot w = (a + ib) \cdot (c + id) = ac - bd + i(ad + bc).$$

$$kz = ka + i(kb), \quad k \in R.$$

$\bar{z} = a - ib$  is called the complex conjugate of  $z = a + ib$ .

$$\overline{z + w} = \bar{z} + \bar{w}, \quad \overline{z - w} = \bar{z} - \bar{w}, \quad \overline{z \cdot w} = \bar{z} \cdot \bar{w}, \quad \overline{\bar{z}} = z,$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, \quad z + \bar{z} = 2 \operatorname{Re} z, \quad z - \bar{z} = 2i \operatorname{Im} z.$$

Let  $z = a + ib$  and  $w = c + id \neq 0$ . Then

$$\begin{aligned} \frac{z}{w} &= \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2} = x + iy. \end{aligned}$$

The absolute value (modulus) of  $z = a + ib$  is

$$|z| = \sqrt{z \bar{z}} = \sqrt{a^2 + b^2}$$

We have the following equalities:

$$|zw| = |z| \cdot |w|, \quad z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

**Example:** Let  $z = 9 - 8i$  and  $w = 5 + 2i$ . Then find  $|z|$ ,  $|w|$ ,  $\left|\frac{z}{w}\right|$ .

Write  $\frac{z}{w}$  in the form of  $a + ib$ .

**Solution:**

$$|z| = \sqrt{9^2 + (-8)^2} = \sqrt{145}$$

$$|w| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{\sqrt{145}}{\sqrt{29}} = \frac{\sqrt{5 \cdot 29}}{\sqrt{29}} = \sqrt{5}$$

$$\frac{z}{w} = \frac{9 - 8i}{5 + 2i} = \frac{9 - 8i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} = \frac{(45 - 16) + i(-40 - 18)}{25 + 4} = 1 - 2i.$$

**Exercises:**

1. Let  $z = 3 + 4i$  and  $w = 5 - 2i$ . Express the followings in the form of  $a + ib$ .

$$(z - w)^2, \quad \frac{z}{w}, \quad \frac{\bar{z}}{\bar{w}}, \quad \frac{1}{z^2}, \quad \frac{w}{2z}.$$

2. Find:  $\operatorname{Re} \frac{1}{2 + i}$ ,  $\operatorname{Im} \frac{2 + i}{3 + 4i}$ ,  $\operatorname{Im} \frac{2 - i}{3 - 4i}$ .

3. Write the followings in the form of  $a + ib$ .

$$\frac{11 + 2i}{4 + 3i}, \quad (3 + 5i)(3 - 5i), \quad \frac{6 + i}{7 + 3i},$$

$$(7 - 3i) - (-2 + 4i), \quad \frac{1}{(3 + 4i)^2}, \quad \frac{\sqrt{3} + i}{(1 - i)(\sqrt{3} - i)}.$$

4. Solve for  $z$  if:

$$iz = 2 - i, \quad (4 - 3i)\bar{z} = i.$$

5. If  $z = 1 - 5i$  and  $w = 3 + 4i$ , find

$$|z|, |w|, |z/w|, |\overline{z/w}|, \text{ and } |\bar{z}/\bar{w}|,$$

### Trigonometric Ratios

$\theta$	radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
$0^\circ$	0	0	1	0
$30^\circ$	$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$
$45^\circ$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60^\circ$	$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$
$90^\circ$	$\pi/2$	1	0	–

### Polar Form of a Complex Number

Let  $z = a + ib$ . Then

$$\cos \theta = \frac{a}{|z|} \implies a = |z| \cos \theta$$

$$\sin \theta = \frac{b}{|z|} \implies b = |z| \sin \theta$$

$$z = a + ib = |z| \cos \theta + i|z| \sin \theta = |z|(\cos \theta + i \sin \theta) = |z|cis\theta,$$

where  $\theta$  is the angle between the positive real axis and the point  $z$ ,  $-\pi < \theta \leq \pi$ .

$\theta$  is called the argument of  $z$ , and denoted by  $\theta = \arg z$ . (All angles are measured in radians). The polar form of  $z$  is

$$z = |z|(\cos \theta + i \sin \theta).$$

**Example:** Find the polar form of  $z = 1 + i$ .

**Solution:**  $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\left. \begin{array}{l} \cos \theta = \frac{a}{|z|} = \frac{1}{\sqrt{2}} \\ \sin \theta = \frac{b}{|z|} = \frac{1}{\sqrt{2}} \end{array} \right\} \implies \theta = \pi/4 \implies z = \sqrt{2}(\cos \pi/4 + i \sin \pi/4).$$

**Example:** What is the polar form of  $z = 3 + i3\sqrt{3}$ ?

**Solution:**  $|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$ .

$$\left. \begin{array}{l} \cos \theta = \frac{a}{|z|} = \frac{3}{6} = \frac{1}{2} \\ \sin \theta = \frac{b}{|z|} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \end{array} \right\} \implies \theta = \pi/3 \implies z = 6(\cos \pi/3 + i \sin \pi/3).$$

**Example:** What is the polar form of  $z = \sqrt{2} - i\sqrt{2}$ ?

**Solution:**  $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = 2$ .

$$\left. \begin{array}{l} \cos \theta = \frac{a}{|z|} = \frac{\sqrt{2}}{2} \\ \sin \theta = \frac{b}{|z|} = \frac{-\sqrt{2}}{2} \end{array} \right\} \implies \theta = -\pi/4 \implies z = 2(\cos \pi/4 - i \sin \pi/4).$$

**Exercises:**

- Write the polar form of the following complex numbers:

$$z = -4 + 4i, \quad z = 4i, \quad z = -7, \quad z = 1, \quad z = \frac{2 + 2i}{1 - i}.$$

- Represent in the form of  $a + ib$ :

$$z = 4(\cos \pi/2 + i \sin \pi/2), \quad z = \sqrt{8}(\cos \pi/4 + i \sin \pi/4),$$

$$z = 2\text{cis}(-\pi/6), \quad z = \frac{2\text{cis}(-3\pi/4)}{2\text{cis}(5\pi/6)}.$$

## Complex Multiplication and Division in Polar Form

Let  $z_1 = |z_1| \operatorname{cis} \theta_1$  and  $z_2 = |z_2| \operatorname{cis} \theta_2$ .

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \operatorname{cis}(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2),$$

$$\overline{z_1} = r_1 \operatorname{cis}(-\theta_1)$$

**Example:** Let  $z = 2 \operatorname{cis} \frac{3\pi}{8}$  and  $w = 5 \operatorname{cis} \frac{2\pi}{3}$ . Then,

$$\overline{z} = 2 \operatorname{cis} \frac{-3\pi}{8} \text{ and } \overline{w} = 5 \operatorname{cis} \frac{-2\pi}{3},$$

$$z \cdot w = \left( 2 \operatorname{cis} \frac{3\pi}{8} \right) \left( 5 \operatorname{cis} \frac{2\pi}{3} \right) = 2 \cdot 5 \operatorname{cis} \left( \frac{3\pi}{8} + \frac{2\pi}{3} \right) = 10 \operatorname{cis} \left( \frac{25\pi}{24} \right),$$

$$\frac{z}{w} = \frac{2}{5} \operatorname{cis} \left( \frac{3\pi}{8} - \frac{2\pi}{3} \right) = \frac{2}{5} \operatorname{cis} \left( \frac{-7\pi}{24} \right).$$

**Example:**  $z = \operatorname{cis} \left( \frac{\pi}{2} \right)$  and  $w = 2 \operatorname{cis} \left( \frac{-\pi}{3} \right)$ .

Find  $z \cdot w$ ,  $\frac{z}{w}$ ,  $\overline{z}$  and  $\overline{w}$  and write them in their standard forms.

**Solution:**

$$z \cdot w = \operatorname{cis} \left( \frac{\pi}{2} \right) \cdot 2 \operatorname{cis} \left( \frac{-\pi}{3} \right) = 2 \operatorname{cis} \left( \frac{\pi}{2} - \frac{\pi}{3} \right)$$

$$= 2 \operatorname{cis} \left( \frac{\pi}{6} \right) = 2 \left( \frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = \sqrt{3} + i.$$

$$\frac{z}{w} = \frac{\operatorname{cis} \left( \frac{\pi}{2} \right)}{2 \operatorname{cis} \left( \frac{-\pi}{3} \right)} = \frac{1}{2} \operatorname{cis} \left( \frac{\pi}{2} - \frac{-\pi}{3} \right)$$

$$= \frac{1}{2} \operatorname{cis} \left( \frac{5\pi}{6} \right) = \frac{1}{2} \left( \frac{-\sqrt{3}}{2} + i \frac{1}{2} \right) = \frac{-\sqrt{3}}{4} + i \frac{1}{4}.$$

$$\overline{z} = \operatorname{cis} \left( \frac{-\pi}{2} \right) = -i.$$

$$\overline{w} = 2 \operatorname{cis} \left( \frac{\pi}{3} \right) = 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = 1 + i\sqrt{3}.$$

## De Moivre's Theorem

Let  $n$  be a positive integer  $n$ . Then

$$z^n = (|z|(\cos \theta + i \sin \theta))^n = |z|^n(\cos n\theta + i \sin n\theta).$$

**Example:** Write  $z = (1 + i)^{20}$  in the form of  $a + ib$ .

**Solution:**  $1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ .

$$\begin{aligned} (1 + i)^{20} &= (\sqrt{2})^{20} \left( \cos \frac{20\pi}{4} + i \sin \frac{20\pi}{4} \right) = 2^{10}(\cos(5\pi) + i \sin(5\pi)) \\ &= 2^{10}(\cos \pi + i \sin \pi) = 2^{10}(-1 + i \cdot 0) = -2^{10} = -1024. \end{aligned}$$

Alternatively, we have

$$(1 + i)^2 = 2i \implies (1 + i)^4 = (2i)^2 = -4.$$

$$(1 + i)^{20} = ((1 + i)^4)^5 = (-4)^5 = -2^{10}.$$

**Exercises:** Express the following complex numbers in the form of  $a + ib$ .

1.  $z = (2cis(\pi/3))^6$ . Ans: 64.
2.  $z = (-1 + i)^4$ .
3.  $z = (1 - i)^{10}$ . Ans:  $-32i$
4.  $z = (1 - i)^{27}$ . Ans:  $-2^{13}(1 + i)$
5.  $z = (1 + i)^{12}$ . Ans:  $-64$
6.  $z = (1 - i)^6(\sqrt{3} + i)^3$ . Ans:  $-64$
7.  $z = (\sqrt{3} - i)^9(2 - 2i)^5$ . Ans:  $-65536(1 + i)$ .

### Roots of a Complex Number

$$z^n = \alpha \operatorname{cis} \theta \implies z_k = \sqrt[n]{\alpha} \operatorname{cis} \left( \frac{\theta + 2k\pi}{n} \right), \text{ where } k = 0, 1, 2, \dots, n-1.$$

**Example:** Find all fourth roots of 1.

**Solution:**

$$z^4 = 1 = \cos 0 + i \sin 0 \implies z_k = \operatorname{cis} \left( \frac{0 + 2k\pi}{4} \right), k = 0, 1, 2, 3.$$

$$z_0 = \operatorname{cis} 0 = 1, \quad z_1 = \operatorname{cis} \frac{\pi}{2} = i, \quad z_2 = \operatorname{cis} \pi = -1, \quad z_3 = \operatorname{cis} \frac{3\pi}{2} = -i.$$

**Example:** Find all fourth roots of  $i$ .

**Solution:**

$$z^4 = i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \implies z_k = \operatorname{cis} \left( \frac{\frac{\pi}{2} + 2k\pi}{4} \right), k = 0, 1, 2, 3.$$

$$z_0 = \operatorname{cis} \left( \frac{\pi}{8} \right), \quad z_1 = \operatorname{cis} \left( \frac{5\pi}{8} \right), \quad z_2 = \operatorname{cis} \left( \frac{9\pi}{8} \right), \quad z_3 = \operatorname{cis} \left( \frac{13\pi}{8} \right).$$

**Example:** Let  $z^3 = -8i$ . Find  $z$  and write it in the standard form.

**Solution:**  $\alpha = |-8i| = 8, \theta = -\pi/2$ .

$$z_k = \sqrt[3]{8} \operatorname{cis} \left( \frac{-\pi/2 + 2k\pi}{3} \right); k = 0, 1, 2.$$

$$z_0 = 2 \operatorname{cis} \left( \frac{-\pi}{6} \right) = 2 \left( \frac{\sqrt{3}}{2} + i \frac{-1}{2} \right) = \sqrt{3} - i.$$

$$z_1 = 2 \operatorname{cis} (\pi/2) = 2(0 + i) = 2i.$$

$$z_2 = 2 \operatorname{cis} (7\pi/6) = 2 \left( \frac{-\sqrt{3}}{2} - i \frac{1}{2} \right) = -\sqrt{3} - i.$$

**Example:** Find the roots of  $z^2 + z + 1 = 0$ .

**Solution:**

$$\begin{aligned} z^2 + z + 1 &= \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0 \implies \left(z + \frac{1}{2}\right)^2 = -\frac{3}{4} \\ \implies z + \frac{1}{2} &= \pm \frac{\sqrt{3}}{2}i \implies z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \end{aligned}$$

**Example:** Find the roots of  $z^2 - 4z + 5 = 0$ .

**Solution:**

$$\begin{aligned} z^2 - 4z + 5 &= (z - 2)^2 + 1 = 0 \implies (z - 2)^2 = -1 \\ \implies z - 2 &= \pm i \implies z = 2 \pm i. \end{aligned}$$

**Exercises:** Find all complex numbers such that:

- |                |                             |
|----------------|-----------------------------|
| 1. $z^2 = i$   | 6. $z^3 = -27i$             |
| 2. $z^3 = i$   | 7. $z^3 = 64i$              |
| 3. $z^3 = -i$  | 8. $z^4 = -1$               |
| 4. $z^3 = -1$  | 9. $z^4 = 2(i\sqrt{3} - 1)$ |
| 5. $z^3 = 27i$ | 10. $z^6 = -64$             |

**Answers:**

- $z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, z_1 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ .
- $z_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2}, z_1 = \frac{-\sqrt{3}}{2} + i\frac{1}{2}, z_2 = -i$ .
- $z_0 = i, z_1 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}, z_2 = \frac{\sqrt{3}}{2} - i\frac{1}{2}$ .
- $z_0 = -1, z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}, z_2 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ .
- $z_0 = -3i, z_1 = -\frac{3\sqrt{3}}{2} + i\frac{3}{2}, z_2 = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$ .
- $z_0 = 3i, z_1 = -\frac{3\sqrt{3}}{2} - i\frac{3}{2}, z_2 = \frac{3\sqrt{3}}{2} - i\frac{3}{2}$ .
- $z_0 = -4i, z_1 = -2\sqrt{3} + 2i, z_2 = 2\sqrt{3} + 2i$ .
- $\sqrt{2} e^{\pi i/6}, \sqrt{2} e^{4\pi i/6}, \sqrt{2} e^{7\pi i/6}, \sqrt{2} e^{10\pi i/6}$ .

Note that  $e^{i\theta} = \cos \theta + i \sin \theta$ .

### Complex Eigenvalues and Complex Eigenvectors

Let  $\lambda \in \mathbb{C}$ . If  $Ax = \lambda x$  for some non-zero vector  $x$  in  $\mathbb{C}^n$ , then  $\lambda$  is called a complex eigenvalue, and  $x$  is called a complex eigenvector.

We note that

$$\det(A - \lambda I) = 0 \iff Ax = \lambda x$$

for some non-zero vector  $x$  in  $\mathbb{C}^n$ .

If  $x$  is a vector in  $\mathbb{C}^n$ , then the vector  $\bar{x}$ , whose entries are the complex conjugates of the entries in  $x$ , is called the complex conjugate of  $x$ .

#### Real matrices with complex eigenvalues:

Let  $A$  be a real  $n \times n$  matrix. Let  $\lambda$  be a complex eigenvalue of  $A$  with a corresponding eigenvector  $x$  in  $\mathbb{C}^n$ . Then  $\bar{\lambda}$  is also an eigenvalue of  $A$  with the corresponding eigenvector  $\bar{x}$ :

$$Ax = \lambda x \iff \overline{Ax} = \overline{\lambda x} \iff \overline{A} \bar{x} = \bar{\lambda} \bar{x} \iff A \bar{x} = \bar{\lambda} \bar{x}$$

Thus the complex eigenvalues of a real matrix  $A$  occur in conjugate pairs.

**Example:** Let  $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ .

- Find the eigenvalues of  $A$ , and a basis for each eigenspace in  $\mathbb{C}^2$ .
- Diagonalize  $A$ .

**Solution:**

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & -2 \\ 1 & 3 - \lambda \end{bmatrix} \implies \det(A - \lambda I) = \lambda^2 - 4\lambda + 5 = 0,$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2} = 2 \pm i.$$

$\lambda_1 = 2 + i$ :

$$A - (2 + i)I = \begin{bmatrix} -1 - i & -2 \\ 1 & 1 - i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 - i \\ 0 & 0 \end{bmatrix}.$$

Thus the solution of  $(A - (2 + i)I)x = 0$  is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 + i \\ 1 \end{bmatrix},$$

$$v_1 = \begin{bmatrix} -1 + i \\ 1 \end{bmatrix}, \quad E_{2+i} = \text{Span} \left\{ \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} \right\}.$$

$\lambda_2 = 2 - i$ :

Since  $\lambda_2 = \overline{\lambda_1}$ , an eigenvector corresponding to  $\lambda_2$  is

$$v_2 = \overline{v_1} = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} \text{ and } E_{2-i} = \text{Span} \left\{ \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} \right\}.$$

b)  $A = PDP^{-1}$ , where

$$P = \begin{bmatrix} -1 + i & -1 - i \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 + i & 0 \\ 0 & 2 - i \end{bmatrix}.$$

**Example:** Let  $A = \begin{bmatrix} 0 & 3 + 4i \\ 3 - 4i & 0 \end{bmatrix}$ .

Find the eigenvalues and corresponding eigenvectors of  $A$ .

**Solution:**

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 + 4i \\ 3 - 4i & -\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \implies \lambda_1 = 5, \quad \lambda_2 = -5.$$

$\lambda_1 = 5$ :

$$A - 5I = \begin{bmatrix} -5 & 3 + 4i \\ 3 - 4i & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3-4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A - 5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_2}{5} \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} \\ v_1 = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} \\ E_5 = \text{Span} \left\{ \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} \right\}. \end{cases}$$

$\lambda_2 = -5$ :

$$A - (-5)I = \begin{bmatrix} 5 & 3 + 4i \\ 3 - 4i & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3+4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\left\{ \begin{array}{l} (A + 5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_2}{5} \begin{bmatrix} -(3 + 4i) \\ 5 \end{bmatrix} \\ v_2 = \begin{bmatrix} -3 - 4i \\ 5 \end{bmatrix} \\ E_2 = \text{Span} \left\{ \begin{bmatrix} -3 - 4i \\ 5 \end{bmatrix} \right\} \end{array} \right.$$

Note that the matrix  $A$  has

- complex entries,
- real eigenvalues,
- complex eigenvectors  $v_1$  and  $v_2$  such that  $v_2 \neq \bar{v}_1$ .

$$A = PDP^{-1} \quad \text{where } P = \begin{bmatrix} 3 + 4i & -3 - 4i \\ 5 & 5 \end{bmatrix} \quad \text{and } D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

**Example:** Let  $A = \begin{bmatrix} i & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$ . If possible, diagonalize  $A$ .

**Solution:**  $\lambda = i$  is the only eigenvalue of  $A$ .

$$A - iI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - iI)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$E_i = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since  $A$  is a  $3 \times 3$  matrix and has only two linearly independent eigenvectors,  $A$  is not diagonalizable.

**Example:** Let  $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$ . If possible, diagonalize  $A$ .

**Solution:**

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 0 & -4 \\ 0 & 1 - \lambda & 0 \\ 2 & 0 & -2 - \lambda \end{vmatrix} = (1 - \lambda)(\lambda^2 + 4) = 0 \implies \lambda = 1, -2i, 2i.$$

$\lambda_1 = 1$ :

$$A - I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\lambda_2 = -2i$ :

$$A + 2iI = \begin{bmatrix} 2 + 2i & 0 & -4 \\ 0 & 1 + 2i & 0 \\ 2 & 0 & -2 + 2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 + i \\ 0 & 1 + 2i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2iI)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 - i \\ 0 \\ 1 \end{bmatrix} \implies v_2 = \begin{bmatrix} 1 - i \\ 0 \\ 1 \end{bmatrix}.$$

$\lambda_3 = 2i$ :

Since  $A$  is a real matrix and  $\lambda_3 = \overline{\lambda_2}$ , an eigenvector for  $\lambda_3$  is

$$v_3 = \overline{v_2} = \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix}.$$

$$A = PDP^{-1}, \text{ where } P = \begin{bmatrix} 0 & 1 - i & 1 + i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}.$$

**Exercise:** Let  $A = \begin{bmatrix} -5 & 6 & 2 \\ -3 & 4 & 1 \\ -5 & 5 & 2 \end{bmatrix}$ .

Find the eigenvalues and corresponding eigenvectors of  $A$ . Diagonalize  $A$ , if possible.

**Answer:** The eigenvalues are  $\lambda_1 = 1$ ,  $\lambda_2 = i$ ,  $\lambda_3 = -i$  with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 - i \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 + i \end{bmatrix}.$$