15 Complex Numbers

15.0.1 Introductory consideration

We can easily solve the equation $x^2 - 4 = 0$. The answer is $x = \pm 2$; in particular, x is a rational number, even an integer. The equation $x^2 - 2 = 0$ is a bit more tricky. The solution $x = \pm\sqrt{2}$ is not a rational number. Instead, we have defined the square root of a positive number as the real number that gives the original number back when multiplied by itself. But what should we do with the equation $x^2 + 1 = 0$? The answer cannot be a real number. (Why?) Can we do the same as above and define a number whose square equals -1? Indeed, this is what mathematicians did in the eighteenth century (it was a daring act and caused a lot of controversy), and they called that number 'i' for *imaginary*. (We will see that complex numbers are hardly more imaginary than $\sqrt{2}$.)

15.1 Definition

A *complex number* z is a number of the form

$$z = a + bi$$

with real numbers a, b and the symbol i that satisfies $i^2 = -1$. We call $a = \mathrm{Re}(z)$ the *real part* of z and $b = \mathrm{Im}(z)$ the *imaginary part* of z. The real number a can be considered the complex number $a + 0i$. A complex number of the form $z = bi$ is called *purely imaginary*.

15.2 Addition, subtraction, and multiplication of complex numbers

Complex numbers are easily added, subtracted and multiplied, if we keep the rule $i^2 = -1$ in mind and use the distributive laws.

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

$$(a + bi) \times (c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i\,,$$

since $i^2 = -1$.

15.2.1 Examples

1. $(3 + 5i) + (2 - 7i) = 5 - 2i$

2. $(0.5 + 1.7i) - (0.8 - 2.6i) = -0.3 + 4.3i$

3. $(-3 + 2i) \times (4 - 5i) = (-12 - (-10)) + (15 + 8)i = -2 + 23i$

4. $(2 - 0.5i) \times (3 + 4i) = (6 - (-2)) + (-1.5 + 8)i = 8 + 6.5i$

5. $(9 + 2i) + 5 = (9 + 2i) + (5 + 0i) = 14 + 2i$

6. $-3i + (2 + 3i) = (0 - 3i) + (2 + 3i) = 2 + 0i = 2$

7. $2 \times (3 - 5i) = 6 - 10i$

8. $3i \times (-1 + 4i) = -12 - 3i$

Before we look at inverses and division of complex numbers, we introduce the *complex conjugate* of a complex number.

15.3 The complex conjugate

The *complex conjugate* of $z = a + bi$ is $\bar{z} = a - bi$, i.e., we simply change the sign of the imaginary part. Since the multiplication

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

always produces a non-negative real number, we can take the square root. We define the *modulus* or *absolute value* of $z = a + bi$ as

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

From the identity $z\bar{z} = |z|^2$, we find the inverse of z to be

$$\frac{1}{z} = z^{-1} = \bar{z}/|z|^2.$$

Example 56. Start with $z = 3+4i$ and $w = 2-i$. The complex conjugates are $\bar{z} = 3-4i$ and $\bar{w} = 2+i$. The absolute values are $|z| = 5$ and $|w| = \sqrt{5}$. The inverses are

$$z^{-1} = \frac{1}{25}(3 - 4i)$$
$$w^{-1} = \frac{1}{5}(2 + i).$$

Finally, we can divide

$$\frac{z}{w} = z\frac{\bar{w}}{|w|^2} = \frac{1}{5}(2 + 11i)$$
$$\frac{w}{z} = w\frac{\bar{z}}{|z|^2} = \frac{1}{25}(2 - 11i).$$

Another way to think about this: make the denominator real (similar to the way you'd rationalize the denominator) by multiplying top and bottom by the conjugate of the denominator (i.e., “real-ize” the denominator). Thus,

$$\begin{aligned}\frac{1}{z} &= \frac{1}{3+4i} \\ &= \frac{1}{3+4i}\cdot\frac{3-4i}{3-4i} \\ &= \frac{3-4i}{3^2-(4i)^2} \\ &= \frac{3-4i}{3^2+4^2} \\ &= \frac{3-4i}{25}\end{aligned}$$

Similarly for $\frac{1}{w}$:

$$\begin{aligned}\frac{1}{w} &= \frac{1}{2+i}\cdot\frac{2-i}{2-i} \\ &= \frac{2-i}{2^2-i^2} \\ &= \frac{2-i}{5}\end{aligned}$$

$$\begin{aligned}
\frac{z}{w} &= \frac{3+4i}{2-i} \cdot \frac{2+i}{2+i} \\
&= \frac{6+3i+8i+4i^2}{2^2+1^2} \\
&= \frac{2+11i}{5} \\
\frac{w}{z} &= \frac{2-i}{3+4i} \cdot \frac{3-4i}{3-4i} \\
&= \frac{6-8i-3i+4i^2}{9+16} \\
&= \frac{2-11i}{25}
\end{aligned}$$

Example 57. Start with $z = 1 - 4i$ and $w = 0.5 + 3i$. The complex conjugates are $\bar{z} = 1 + 4i$ and $\bar{w} = 0.5 - 3i$. The absolute values are $|z| = \sqrt{17}$ and $|w| = \sqrt{37/4}$. The inverses are

$$\begin{aligned}
z^{-1} &= \frac{1}{17}(1+4i) \\
w^{-1} &= \frac{4}{37}(0.5-3i).
\end{aligned}$$

Division gives

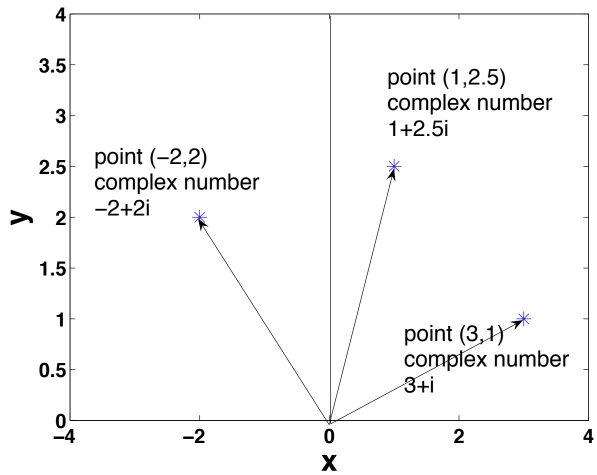
$$\begin{aligned}
\frac{z}{w} &= z\frac{\bar{w}}{|w|^2} = \frac{4}{37}(-11.5-5i) \\
\frac{w}{z} &= w\frac{\bar{z}}{|z|^2} = \frac{1}{17}(-11.5+5i).
\end{aligned}$$

Alternatively, we can "real-ize" the denominator as before. Thus $\frac{z}{w}$ and $\frac{w}{z}$ are

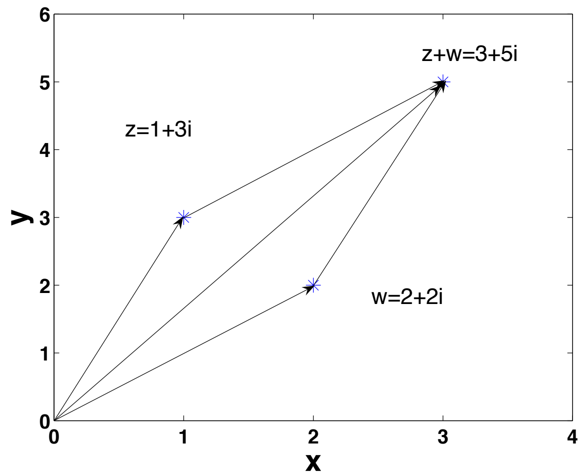
$$\begin{aligned}
\frac{z}{w} &= \frac{1-4i}{0.5+3i} \cdot \frac{0.5-3i}{0.5-3i} \\
&= \frac{0.5-3i-2i+12i^2}{0.25+9} \\
&= \frac{-11.5-5i}{9.25} \\
\frac{w}{z} &= \frac{0.5+3i}{1-4i} \cdot \frac{1+4i}{1+4i} \\
&= \frac{0.5+2i+3i-12}{1+16} \\
&= \frac{-11.5+5i}{17}
\end{aligned}$$

15.4 Geometric interpretation

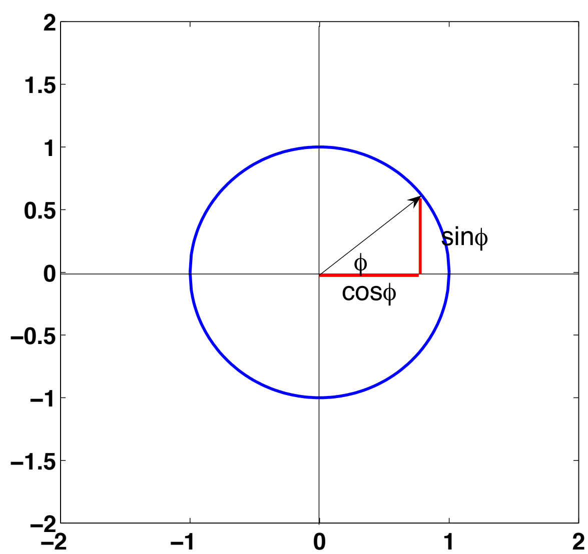
It is very helpful to think of a complex number as a point in the plane with the real part as the x-value and the imaginary part as the y-value. Hence, we identify the complex number $z = a + bi$ with the point (a, b) or with the vector (arrow) from the origin to the point (a, b). (We will talk about vectors in more detail shortly). Then the absolute value of the complex number is simply the distance of the corresponding point from the origin or the length of the vector (arrow).



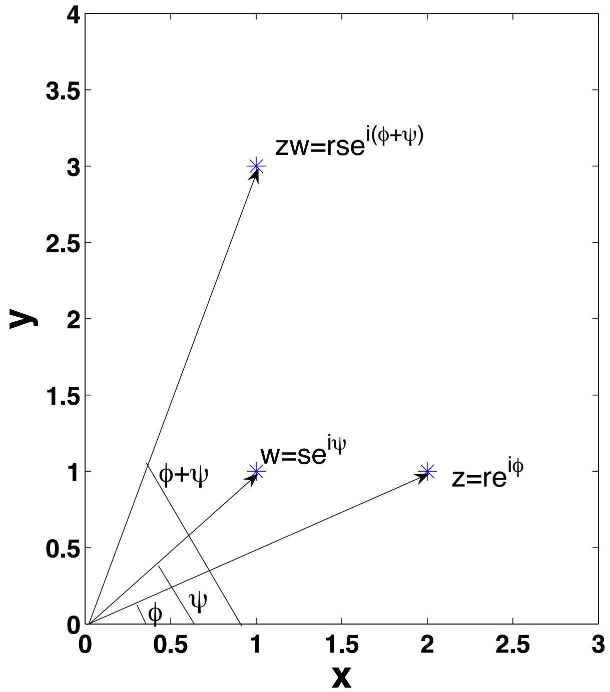
With this correspondence, the addition of complex numbers become the addition of vectors as it is known from the physics of forces.



To interpret multiplication, we take a slightly different point of view. Instead of giving the coordinates of the vector as the endpoint (a, b), we consider its length $r \geq 0$ and the angle ϕ it makes with the x-axis (counterclockwise) as $(r \cos \phi, r \sin \phi)$. This representation is called *polar coordinates.*



Then multiplication of two numbers is simply multiplication of the lengths and addition of the angles.



We write

$$z = r(\cos\phi + i\sin\phi) \quad \text{and} \quad w = s(\cos\psi + i\sin\psi).$$

Then we multiply, using the trigonometric identities

$$\begin{aligned} zw &= r(\cos\phi + i\sin\phi) \times s(\cos\psi + i\sin\psi) \\ &= rs[\cos\phi\cos\psi - \sin\phi\sin\psi + i(\cos\phi\sin\psi + \cos\psi\sin\phi)] \\ &= rs[\cos(\phi+\psi) + i\sin(\phi+\psi)]. \end{aligned}$$

15.5 Observation and definition

Every complex number of the form $z = \cos\phi + i\sin\phi$ has absolute value one. We introduce the exponential notation (known as Euler's formula)

$$\exp(\mathrm{i}\phi) = e^{i\phi} = \cos\phi + i\sin\phi.$$

It might look strange at first, but the same rules as for the real exponential function apply. In fact, if we denote

$$f(\phi) = \cos\phi + i\sin\phi$$

then

$$\begin{aligned} f'(\phi) &= -\sin\phi + i\cos\phi \\ \frac{f'(\phi)}{f(\phi)} &= \frac{-\sin\phi + i\cos\phi}{\cos\phi + i\sin\phi} \\ &= \frac{-\sin\phi + i\cos\phi}{\cos\phi + i\sin\phi} \cdot \frac{\cos\phi - i\sin\phi}{\cos\phi - i\sin\phi} \\ &= \frac{-\sin\phi\cos\phi + i\sin^2\phi + i\cos^2\phi + \cos\phi\sin\phi}{\cos^2\phi + \sin^2\phi} \\ &= i \end{aligned}$$

since $\cos^2\phi + \sin^2\phi = 1$. Now integrate:

$$\int \frac{f'(\phi)}{f(\phi)} d\phi = \int i d\phi$$
$$\ln f(\phi) = i\phi$$
$$f(\phi) = e^{i\phi}$$

and thus $\cos\phi + i\sin\phi = e^{i\phi}$.

This has many advantages. First of all, we can write any complex number in polar coordinates as $z = re^{i\phi}$. And we can easily multiply complex numbers in this form. For example, the calculation above becomes a single step (no need to look up the trig identities)

$$re^{i\phi} \times se^{i\psi} = rse^{i(\phi+\psi)}.$$

Example 58.

1. The complex number $z = 1 + i$ has modulus $|z| = \sqrt{2}$ and angle $\phi = \pi/4$. Hence $z = 1 + i = \sqrt{2}e^{i\pi/4}$.
2. The complex number $w = \sqrt{3} + i$ has modulus $|w| = 2$ and angle $\phi = \pi/6$.
   Hence $w = \sqrt{3} + i = 2e^{i\pi/6}$.
3. Their product is $zw = (\sqrt{3} - 1) + (\sqrt{3} + 1)i = 2\sqrt{2}e^{i5\pi/12}$.
4. In general, if $z = a + bi$ then $r = |z| = \sqrt{a^2 + b^2}$. The argument ϕ is not uniquely defined. If we restrict it between $-\pi$ and π the we get

$$\begin{cases} \phi = \arctan(b/a) & \text{if} \quad a > 0 \\ \phi = \arctan(b/a) + \pi & \text{if} \quad a < 0, b > 0 \\ \phi = \arctan(b/a) - \pi & \text{if} \quad a < 0, b < 0 \end{cases}$$

16 Linear Algebra

Example 59. Find equilibria of

$$x' = 2x + y$$
$$y' = 3x + 4y$$

Solution :

$2x + y = 0 \qquad R_1$

$3x + 4y = 0 \qquad R_2$

$8x + 4y = 0 \qquad R_1 \to R_1$ x 4

$3x + 4y = 0$

$8x + 4y = 0$

$-5x \quad = 0 \qquad R_2 \to R_2 - R_1$

$\Rightarrow \; x = 0 \qquad 8(0) + 4y = 0 \Rightarrow \; y = 0$

There is one solution.

Example 60. Find equilibria of

$$x' = 6x - 5y + 4z - 5$$
$$y' = 4x - 4y + 3z - 2$$
$$z' = 6x - 7y + 5z - 3$$

Solution :

$$\begin{aligned} 6x - 5y + 4z &= 5 \qquad R_1 \\ 4x - 4y + 3z &= 2 \qquad R_2 \\ 6x - 7y + 5z &= 3 \qquad R_3 \end{aligned}$$

$$\begin{aligned} 6x - 5y + 4z &= 5 \\ 4x - 4y + 3z &= 2 \\ -2y + z &= -2 \qquad R_3 \to R_3 - R_1 \end{aligned}$$

$$\begin{aligned} 6x + 3y \quad &= 13 \qquad R_1 \to R_1 - 4R_3 \\ 4x + 2y \quad &= 8 \qquad R_2 \to R_2 - 3R_3 \\ -2y + 2 &= -2 \end{aligned}$$

$$\begin{aligned} 6x + 3y \quad &= 13 \\ 2x + y \quad &= 4 \qquad R_2 \to R_2/2 \\ -2y + 2 &= -2 \end{aligned}$$

$$\begin{aligned} 0 \quad &= 1 \qquad R_1 \to R_1 - 3R_2 \\ 2x + y \quad &= 4 \\ -2y + z &= -2 \end{aligned}$$

$\therefore$ no solution.

Example 61. Solve:

$$\begin{aligned} -x + 3y - z &= -4 \qquad R_1 \\ 3x - 8y + 5z &= 14 \qquad R_2 \\ 2x - 6y + 2z &= 8 \qquad R_3 \end{aligned}$$

$$\begin{aligned} -x + 3y - z &= -4 \\ y + 2z &= 2 \qquad R_2 \to R_2 + 3R_1 \\ 0 &= 0 \qquad R_3 \to R_3 + 2R_1 \end{aligned}$$

Since we have more variables than equations , we can let z be any number, ie $z = t, t \in \mathbb{R}$

$$\begin{aligned} y + 2t = 2 \Rightarrow y = 2 - 2t \\ -x + 3(2 - 2t) - t = -4 \\ x = 4 + 6 - 6t - t = 10 - 7t \end{aligned}$$

Therefore, there are infinitely many solutions.

16.1 Shorthand notation: Matrices

A matrix is a rectangular array of numbers. We have m rows and n columns, yielding an m x n matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

We can rewrite a system of equations in matrix form:

$$\begin{array}{llll} -x & +3y & -z & = -4 \\ 3x & -8y & +5z & = 14 \\ 2x & -6y & +2z & = 8 \end{array} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{square matrix}}}{\begin{bmatrix} -1 & 3 & -1 \\ 3 & -8 & 5 \\ 2 & -6 & 2 \end{bmatrix}} \underset{\substack{\uparrow}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{column vectors}}}{\begin{bmatrix} -4 \\ 14 \\ 8 \end{bmatrix}} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{augmented matrix}}}{\left[\begin{array}{ccc|c} -1 & 3 & -1 & -4 \\ 3 & -8 & 5 & 14 \\ 2 & -6 & 2 & 8 \end{array}\right]}$$

Example 62. Solve

$$5x + 6y = 13$$
$$x + 2y = 2$$

$$\left[\begin{array}{cc|c} 5 & 6 & 13 \\ 1 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & -4 & 3 \\ 1 & 2 & 2 \end{array}\right] \quad R_1 \to R_1 - 5R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -4 & 3 \end{array}\right] \quad R_1 \Leftrightarrow R_2$$

This matrix is upper triangular.

$$-4y = 3 \Rightarrow y = -\frac{3}{4}$$

$$x + 2y = 2$$

$$x + 2\left(-\frac{3}{4}\right) = 2$$

$$x - \frac{3}{2} = 2$$

$$x = 2 + \frac{3}{2} = \frac{7}{2}$$

Example 63. Solve

$$x + y + z = 6$$
$$2x + y - z = -6$$
$$3x + 2y - 5z = -35$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & -6 \\ 3 & 2 & -5 & -35 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -18 \\ 0 & -1 & -8 & -53 \end{array}\right] \qquad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -5 & -35 \end{array}\right] \qquad \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\begin{aligned} &-5z = -35 && \Rightarrow z = 7 \\ &-y - 3z = -18 \\ &-y - 3(7) = -18 \end{aligned}$$

$$\begin{aligned} &-y - 21 = -18 \\ &-y = 3 && \Rightarrow y = -3 \\ &x + y + z = 6 \\ &x - 3 + 7 = 6 \\ &x = 6 - 4 && \Rightarrow x = 2 \end{aligned}$$

Example 64. Solve

$$\begin{aligned} x - y - 2z &= 3 \\ -2x - y + z &= 4 \\ 3x \quad - 3z &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ -2 & -1 & 1 & 4 \\ 3 & 0 & -3 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & -3 & -3 & 10 \\ 0 & 3 & 3 & -10 \end{array}\right] \qquad \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & -3 & -3 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\begin{aligned} &z = t && \Rightarrow z = t \\ &-3y - 3z = 10 \\ &-3y - 3t = 10 \\ &-3y = 10 + 3t && \Rightarrow y = -\frac{10}{3} - t \\ &x - y - 2z = 3 \\ &x - \left(-\frac{10}{3} - t\right) - 2t = 3 && \Rightarrow x = -\frac{1}{3} + t \end{aligned}$$

17 Matrices

Two matrices A and B are equal if and only if all their entries are equal.

17.1 Matrix addition

Matrix addition is performed by adding individual entries.

Example 65. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 10 & 19 & -8 \\ 3 & -1 & -15 \\ 2 & -2 & 0 \end{bmatrix}$

$$A + B = \begin{bmatrix} 11 & 21 & -5 \\ 7 & 4 & -9 \\ 9 & 6 & 9 \end{bmatrix}$$

17.2 Multiplication of matrices by a constant

Multiplying a matrix A by a constant c multiplies each entry.

Example 66. $-3B = \begin{bmatrix} -30 & -57 & 24 \\ -9 & 3 & 45 \\ -6 & 6 & 0 \end{bmatrix}$

Example 67. $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\qquad N = \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix}$ $\qquad P = \begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix}$

Find M+2N-3P.

$$\begin{aligned} M + 2N - 3P &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -10 & -12 \\ -14 & -16 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 3 & -27 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 10 - 6 & 2 - 12 \\ 3 - 14 + 3 & 4 - 16 - 27 \end{bmatrix} \\ &= \begin{bmatrix} -15 & -10 \\ -8 & -39 \end{bmatrix} \end{aligned}$$

17.3 Matrix Multiplication

Matrix multiplication: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Example 68.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ -7 & -1 \\ -2 & 0 \end{bmatrix} &= \begin{bmatrix} 1(5) + 2(-7) + 3(-2) & 1(6) + 2(-1) + 3(0) \\ 4(5) + 5(-7) + 6(-2) & 4(6) + 5(-1) + 6(0) \end{bmatrix} \\ &= \begin{bmatrix} 5 - 14 - 6 & 6 - 2 \\ 20 - 35 - 12 & 24 - 5 \end{bmatrix} \\ &= \begin{bmatrix} -15 & 4 \\ -27 & 19 \end{bmatrix} \end{aligned}$$

Important: The number of columns of the first matrix must match the number of rows of the second matrix.

Example 69.

$$NP = \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} -10+6 & -54 \\ -14+8 & -72 \end{bmatrix} = \begin{bmatrix} -4 & -54 \\ -6 & -72 \end{bmatrix}$$

$$PN = \begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix} \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix} = \begin{bmatrix} -10 & -12 \\ 5-63 & 6-72 \end{bmatrix} = \begin{bmatrix} -10 & -12 \\ -58 & -66 \end{bmatrix}$$

$\therefore NP \neq PN \qquad \therefore$ order is important.

Example 70.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 & 19 & -8 \\ 3 & -1 & -15 \\ 2 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10+6+6 & 19-2-6 & -8-30 \\ 40+15+12 & 76-5-12 & -32-75 \\ 70+24+18 & 133-8-18 & -56-120 \end{bmatrix} \\ &= \begin{bmatrix} 22 & 11 & -38 \\ 67 & 59 & -107 \\ 112 & 107 & -176 \end{bmatrix} \end{aligned}$$

Exercise: Find BA. Show that $AA^2 = A^2A = A^3$.

17.4 The Identity Matrix

The identity matrix I is an $n \times n$ matrix with 1's down the diagonal and 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

For any matrix A, $AI = IA = A$.

17.5 Determinants

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the determinant of A is $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

Example 71. $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1(4) - 2(3) = 4 - 6 = -2.$

Example 72. Find $\det(P - N)$

$$\begin{aligned} \det(P-N) &= \det\left(\begin{bmatrix} 2 & 0 \\ -1 & 9 \end{bmatrix} - \begin{bmatrix} -5 & -6 \\ -7 & -8 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} 7 & 6 \\ 6 & 17 \end{bmatrix} \\ &= 7(17) - 6(6) \\ &= 83 \end{aligned}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.
(Look at the pattern, don't memorise the formula.)

Example 73. Find $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= 1(5)(9) + 2(6)(7) + 3(4)(8) - 3(5)(7) - 1(6)(8) - 2(4)(9) \\ &= 45 + 84 + 96 - 105 - 48 - 72 \\ &= 0 \end{aligned}$$

Example 74. Find $\det \begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix}$.

$$\begin{aligned} \det \begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix} &= 8(3)(0) + 2(5)(-4) + 0(1)(0) - 0(3)(-4) - 8(1)(5) - 0(2)(0) \\ &= -40 - 40 \\ &= -80 \end{aligned}$$

17.6 Inverse of a matrix

Suppose A is an $n \times n$ square matrix. If there exists an $n \times n$ matrix B such that $AB = BA = I$, then B is called the inverse of A and is denoted A^{-1}.

Theorem 17.1. *A^{-1} exists if and only if* $\det A \neq 0$.

Therefore $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not invertible, but $\begin{bmatrix} 8 & 2 & 0 \\ 0 & 3 & 5 \\ -4 & 1 & 0 \end{bmatrix}$ is.

To find the inverse:
1. Write the augmented matrix $[A|I]$
2. Row reduce until the matrix is $[I|B]$
3. The matrix B is the inverse of A

Example 75. Find the inverse of $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right] \qquad R_1 - 2R_2$$

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & -2 \\ 1 & 0 & 3 & -5 \end{array}\right] \qquad R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} 0 & 1 & -1 & 2 \\ 1 & 0 & 3 & -5 \end{array}\right] \qquad R_1 \times (-1)$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] \qquad R_1 \Leftrightarrow R_2$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Example 76. Find the inverse of $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & -2 \\ 1 & 3 & 0 & 1 \end{array}\right] \qquad R_1 - 2R_2$$

Since the first row consists of only 0s, we can never obtain the identity matrix on the left hand side. Therefore we can't find the inverse.

Check: $\det \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} = 2(3) - 6(1) = 0 \quad \therefore$ the inverse doesn't exist.

Theorem 17.2. *If* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $a_{11}a_{22} - a_{12}a_{21} \neq 0$ *then* A *is invertible and*

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Note: This only works for 2 x 2 matrices. The denominator is the determinant.

Exercise: Check this for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

Example 77. Find the inverse of $\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right] \qquad \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 1 \end{array}\right] \qquad R_1 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad R_3/(-2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \qquad \begin{array}{c} R_1 - 2R_3 \\ R_2 - 3R_3 \end{array}$$

$$\therefore \text{ the inverse is } \begin{bmatrix} 0 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

17.7 Transpose of a matrix

The transpose of a matrix swaps its rows and its columns.

Example 78. Find the transpose of $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 6 \\ -1 & 1 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 3 & 6 & -1 \end{bmatrix}$$

Example 79. Find the transpose of $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

17.8 Solving linear equations

Example 80. Solve

$$\begin{aligned} 2x + 5y &= 4 \\ x + 3y &= -1 \end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 3(4) - 5(-1) \\ -1(4) + 2(-1) \end{bmatrix} \\
&= \begin{bmatrix} 17 \\ -6 \end{bmatrix} \\
&\therefore\ x = 17,\ y = -6
\end{aligned}$$

Therefore if A^{-1} exists then the solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

Example 81. Solve

$$\begin{aligned}
x - y - z &= -1 \\
2x - y + z &= 2 \\
-x + y - z &= 1
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 + 1 \\ \frac{1}{2} + 2 + \frac{3}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \\
&\therefore\ x = 3,\ y = 4,\ z = 0
\end{aligned}$$

18 Eigenvalues and Eigenvectors

If A is a square matrix and $\vec{x}$ is a vector, then $A\vec{x}$ is also a vector. Could it be possible that $A\vec{x}$ is a scalar multiple of $\vec{x}$, say $\lambda\vec{x}$?

If so $\quad A\vec{x} = \lambda\vec{x}$

$\quad A\vec{x} - \lambda\vec{x} = \vec{0} \quad$ (the zero vector)

We can't factor out $\vec{x}$ because $(A - \lambda)\vec{x}$ makes no sense. (How do you substract a number from a matrix?)
But I comes to the rescue! Recall, $I\vec{x} = \vec{x}$.
$\therefore\ \lambda\vec{x} = \lambda I\vec{x}$ if we want.

$$\begin{aligned} A\vec{x} - \lambda\vec{x} &= \vec{0} \\ A\vec{x} - \lambda I\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \qquad \text{this is fine since } \lambda I \text{ is a square matrix.} \end{aligned}$$

If $\vec{x} = \vec{0}$ then this is trivial. So let's suppose that $\vec{x} \neq \vec{0}$.
If $A - \lambda I$ is invertible, then we could take the inverse $\vec{x} = (A - \lambda I)^{-1}\vec{0} = \vec{0}$ (anything multiplied by $\vec{0}$ is $\vec{0}$.)
But this is no good, as we don't want $\vec{x} = \vec{0}$.
$\therefore\ A - \lambda I$ is not invertible. ie. $\det(A - \lambda I) = 0$. This gives us a formula for λ that's independent of $\vec{x}$.
The values λ are called eigenvalues and the vectors $\vec{x}$ are called eigenvectors.

Example 82. Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(2-\lambda) - (2)(3) \\ &= \lambda^2 - \lambda - 2\lambda + 2 - 6 \\ &= \lambda^2 - 3\lambda - 4 = 0 \\ (\lambda - 4)(\lambda + 1) &= 0 \\ \lambda = 4,\ \lambda &= -1 \text{ are the eigenvalues.} \end{aligned}$$

To find eigenvectors, put into the equation $(A - \lambda I)\vec{x} = \vec{0}$.

$\lambda = 4:$

$$\begin{aligned} (A - 4I)\vec{x} &= \vec{0} \\ \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} -3x_1 + 2x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned} \qquad \text{But these two equations are fundamentally the same.}$$

Note: this will always be true.

We are left with one equation and two variables: $3x_1 - 2x_2 = 0$

$$\therefore \; let \; x_2 = t$$
$$x_1 = \frac{2}{3}t$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} t, \qquad t \in \mathbb{R}, \; t \neq 0.$$

$\lambda = -1$:

$$(A - (-1)I)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1-(-1) & 2 \\ 3 & 2-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{array}{rcl} 2x_1 + 2x_2 & = & 0 \\ 3x_1 + 3x_2 & = & 0 \end{array} \Rightarrow x_1 + x_2 = 0$$
$$let \; x_2 = s, \; x_1 = -s$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s, \qquad s \in \mathbb{R}, s \neq 0.$$

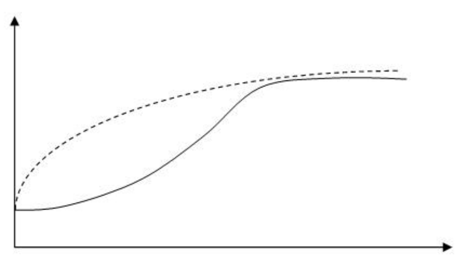
Note: It's crucial to specify that $t \neq 0$ and $s \neq 0$, because the cases $t = 0$ and $s = 0$ correspond to the zero vector, which cannot be an eigenvector.

18.0.1 Algorithm for eigenvalues and eigenvectors

- Use $\det(A - \lambda I) = 0$ to find a polynomial of degree n
- Solve over the complex plane for n eigenvalues
- Use $(A - \lambda I)\vec{x} = \vec{0}$ to solve for $\vec{x}$
- You must get a row of zeroes
- The free variable cannot equal zero.

18.0.2 Biological interpretation of eigenvalues and eigenvectors

- If all $\lambda_i < 0$ then the disease is eliminated.
- If any $\lambda_i > 0$ then the disease propagates.
- $\vec{x}$ is a measure of the speed of propagation.
- Control measures aim to reduce λ_{max} below 0.



Similar λ's, different $\vec{x}$'s

Example 83. Find the eigenvalues and eigenvectors of $B = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$

Eigenvalues:

$$\begin{aligned}
\det(B-\lambda I) &= \det\begin{bmatrix}1-\lambda & 4\\ 1 & -2-\lambda\end{bmatrix}\\
&= (1-\lambda)(-2-\lambda)-4\\
&= \lambda^2-\lambda+2\lambda-2-4\\
&= \lambda^2+\lambda-6\\
&= (\lambda-2)(\lambda+3)=0\\
\lambda &= 2, -3
\end{aligned}$$

Eigenvectors:

$\lambda = 2:$

$$\begin{aligned}
\begin{bmatrix}1-2 & 4\\ 1 & -2-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}-1 & 4\\ 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
x_1-4x_2 &= 0\\
x_2 &= t\\
x_1 &= 4t\\
\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}4\\1\end{bmatrix}t, \qquad t\in\mathbb{R},\ t\neq 0
\end{aligned}$$

$\lambda = -3:$

$$\begin{aligned}
\begin{bmatrix}1-(-3) & 4\\ 1 & -2-(-3)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
x_1+x_2 &= 0\\
x_2 &= s\\
x_1 &= -s\\
\begin{bmatrix}x_1\\x_2\end{bmatrix} &= \begin{bmatrix}-1\\1\end{bmatrix}s, \qquad s\in\mathbb{R},\ s\neq 0
\end{aligned}$$

Example 84. Find the eigenvalues and eigenvectors of $C = \begin{bmatrix}-1 & 1 & 8\\ 5 & 3 & -1\\ 0 & 0 & 9\end{bmatrix}$

Eigenvalues:

$$\begin{aligned}
\det(C-\lambda I) &= \det\begin{bmatrix}-1-\lambda & 1 & 8\\ 5 & 3-\lambda & -1\\ 0 & 0 & 9-\lambda\end{bmatrix}\\
&= (-1-\lambda)(3-\lambda)(9-\lambda)+0+0-0-0-5(1)(9-\lambda)\\
&= (9-\lambda)[(-1-\lambda)(3-\lambda)-5]\\
&= (9-\lambda)[\lambda^2+\lambda-3\lambda-3-5]\\
&= (9-\lambda)(\lambda^2-2\lambda-8)\\
&= (9-\lambda)(\lambda-4)(\lambda+2)=0\\
\lambda &= 9,\ 4,\ -2
\end{aligned}$$

Eigenvectors:

$$\lambda = 9: \quad \begin{bmatrix} -1-9 & 1 & 8 \\ 5 & 3-9 & -1 \\ 0 & 0 & 9-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -10 & 1 & 8 \\ 5 & -6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -11 & 6 \\ 5 & -6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_1 + 2R_2$$

$$-11x_2 + 6x_3 = 0$$

$$5x_1 - 6x_2 - x_3 = 0$$

$$x_3 = t$$

$$x_2 = \frac{6}{11}t$$

$$5x_1 - 6\left(\frac{6}{11}t\right) - t = 0$$

$$x_1 = \frac{47}{55}t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{47}{55} \\ \frac{6}{11} \\ 1 \end{bmatrix} t, \qquad t \in \mathbb{R},\ t \neq 0$$

$$\lambda = 4: \quad \begin{bmatrix} -1-4 & 1 & 8 \\ 5 & 3-4 & -1 \\ 0 & 0 & 9-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 8 & 0 \\ 5 & -1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -5 & 1 & 8 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \qquad R_2 + R_1$$

$$x_3 = 0$$

$$-5x_1 + x_2 + 8x_3 = 0$$

$$x_2 = s$$

$$-5x_1 + s + 8(0) = 0$$

$$x_1 = \frac{1}{5}s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 1 \\ 0 \end{bmatrix} s, \qquad s \in \mathbb{R},\ s \neq 0$$

$$\lambda = -2: \qquad \begin{bmatrix} -1-(-2) & 1 & 8 \\ 5 & 3-(-2) & -1 \\ 0 & 0 & 9-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 8 & 0 \\ 5 & 5 & -1 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 8 & 0 \\ 0 & 0 & -41 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right] \qquad R_2 - 5R_1$$

$$x_3 = 0$$

$$x_1 + x_2 + 8x_3 = 0$$

$$x_2 = r$$

$$x_1 + r = 0 \to x_1 = -r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} r, \qquad r \in \mathbb{R},\ r \neq 0$$

Exercise. Find the eigenvalues and eigenvectors of

$$D = \begin{bmatrix} 0 & -8 & 6 \\ 2 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix} \qquad\qquad E = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

18.1 Finding general solutions of linear systems

To find the general solution of $\begin{bmatrix} x' \\ y' \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}$, let's start with a solution of the following form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \lambda e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = J e^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = J \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This implies λ is an eigenvalue of J with corresponding eigenvector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

The general form for 2 x 2 matrices is: $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Example 85. Solve the following system with initial conditions $x(0) = 5, y(0) = 1$.

$$x' = -6x + 9y$$
$$y' = 3x$$

$$J = \begin{bmatrix} -6 & 9 \\ 3 & 0 \end{bmatrix}$$

$$\begin{aligned}
\det(J - \lambda I) &= \det \begin{bmatrix} -6-\lambda & 9 \\ 3 & -\lambda \end{bmatrix} \\
&= (6+\lambda)\lambda - 27 \\
&= \lambda^2 + 6\lambda - 27 \\
&= (\lambda+9)(\lambda-3) \\
&\rightarrow \lambda = -9, 3
\end{aligned}$$

$$\lambda_1 = -9 : \begin{bmatrix} -6+9 & 9 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ 3 & 9 & 0 \end{array}\right]$$

$$v_1 + 3v_2 = 0$$
$$v_2 = r$$
$$v_1 = -3r$$

$$\therefore \textit{the eigenvectors are given by } \begin{bmatrix} -3 \\ 1 \end{bmatrix} r, r \in \mathbb{R}, r \neq 0.$$

$$\lambda_2 = 3 : \begin{bmatrix} -6-3 & 9 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -9 & 9 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$w_1 - w_2 = 0$$
$$w_2 = s$$
$$w_1 = s$$

$$\therefore \textit{the eigenvectors are given by } \begin{bmatrix} 1 \\ 1 \end{bmatrix} s, s \in \mathbb{R}, s \neq 0.$$

So we have $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-9t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Now we use the initial conditions to find the values of c_1 and c_2.

$$-3c_1 + c_2 = 5$$
$$c_1 + c_2 = 1$$
$$\Rightarrow c_1 = -1, c_2 = 2$$

The solution is given by $\begin{bmatrix} x \\ y \end{bmatrix} = -e^{-9t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Exercise: Show that this solution satisfies both the ODEs and the initial condition.

Exercise: Solve

$$x' = -8y + 10z$$
$$y' = -2x + 8z$$
$$z' = -2z$$

with $x(0) = -45$, $y(0) = 7$ and $z(0) = 9$.

19 Multivariable Calculus

Real-valued functions

Almost all biological processes depend on more than one variable.

Example 86. The ambient temperature you need to survive is:

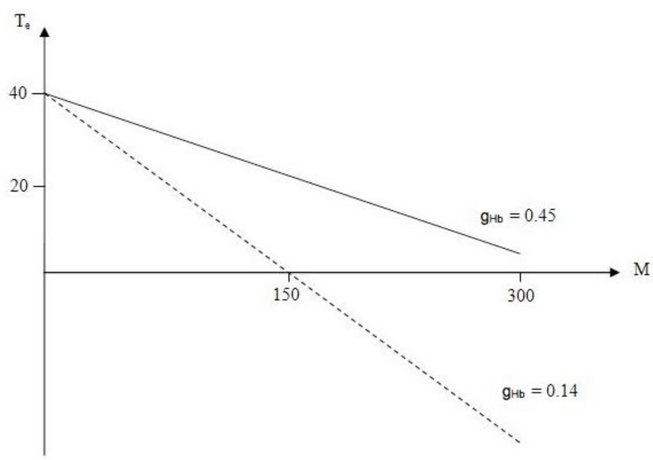
$$T_e = 36 - \frac{(0.9M - 12)(g_{Hb} + 0.95)}{27.8 g_{Hb}}$$

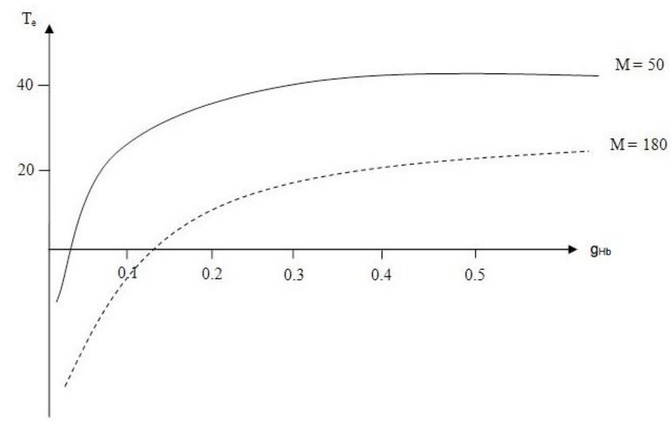
Where T_e is the temperature in degrees celcius
M is metabolic heat
g_{Hb} is thermal conduction, describing how quickly heat is lost.

$g_{Hb} = 0.45\ mol\ m^{-2}s^{-1}$	without clothing
$= 0.14$	for a wool suit
$= 0.04$	for a sleeping bag

$M = 50\ wm^{-2}$	sleeping
$= 95$	writing at a desk
$= 180$	walking at 2.5 mph
$= 350$	walking at 3.5 mph with a 40 lb pack

To meet the required temperature, we can either change M (moving when we get cold) or g_{Hb} (putting on more clothing) or both.





Definition 19.1. *A real-valued function f assigns a real number to an n-dimensional input. We write $f : \mathbb{R}^n \to \mathbb{R}$ and f has the form $f(x_1, x_2, \ldots, x_n) = w$.*

Example 87. $f(x, y) = x + y$. Evaluate f at the points $(-1, 1)$ and $(2, -5)$.

$$f(-1, 1) = -1 + 1 = 0$$
$$f(2, -5) = 2 - 5 = -3$$

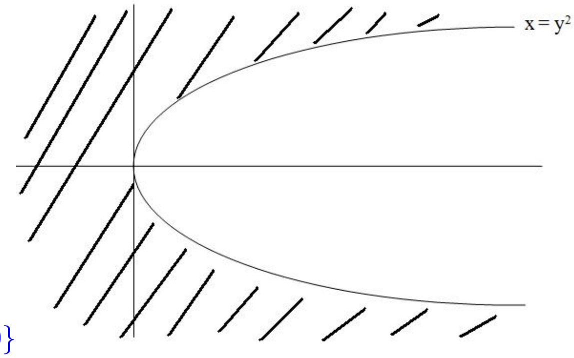
Example 88. Find the domain and range of the function $f(x, y, z) = \frac{xy}{z^2}$.

Domain: $z \neq 0$ $\therefore$ domain is $\{(x, y, z) \in \mathbb{R}^3,\ z \neq 0\}$
Range: $\mathbb{R}$ (since f can take any value).

Example 89. Find the domain and range of $f(x, y) = \sqrt{y^2 - x}$. Sketch the domain in $\mathbb{R}^2$.

$Domain: \ y^2 - x \geqslant 0 \qquad \rightarrow \ y^2 \geqslant x$

$x \leqslant y^2$

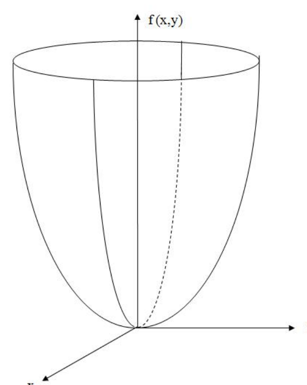


$Range: \{f \in \mathbb{R}: \ f \geqslant 0\}$

Example 90. Graph the function $f(x, y) = x^2 + y^2$

$Domain: \ \mathbb{R}^2$

$Range: \ \{f : f \geqslant 0\}$



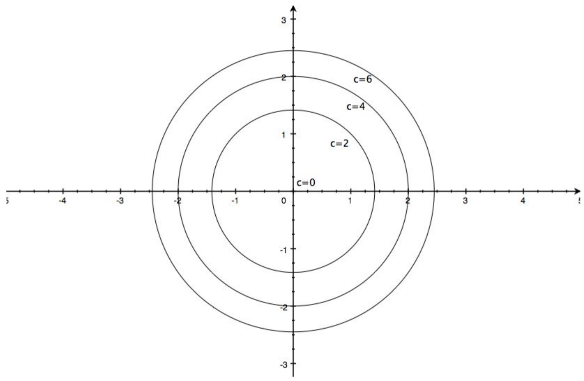
The graph is a parabolic bowl.

Another way to visualize functions is with level curves. We see these in weather maps all the time.

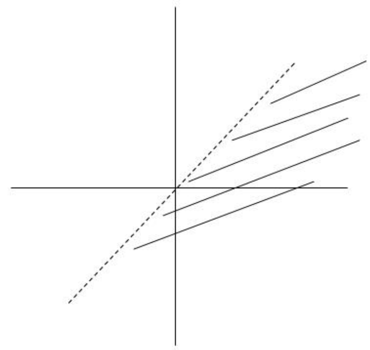
Definition 19.2. *Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. The level curves of f comprise the set of (x, y) points where $f(x, y) = c$.*

Level curves are a "snapshot" of the function, taken at constant values c. It's a way of seeing 3 dimensions but only drawing 2.

Example 91. $f(x, y) = x^2 + y^2$ Plot the level curves for $c = 0, 2, 4, 6$.
$x^2 + y^2 = c$ describes a circle with centre $(0, 0)$ and radius $\sqrt{c}$. (If $c = 0$, this is simply a point)
As c increases, the circles get closer together.



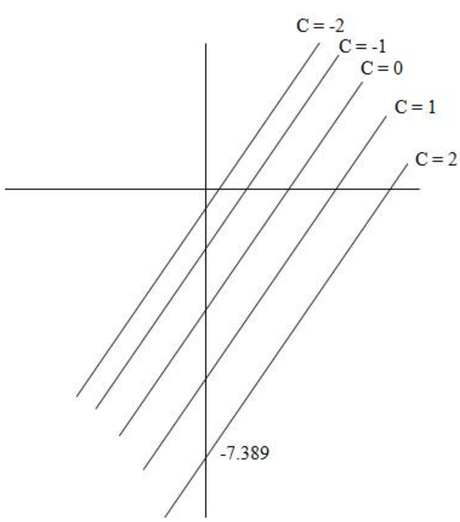
Example 92. $f(x,y) = \ln(x-y)$. Find the domain and range. Sketch the domain and level curves corresponding to $c = -2, -1, 0, 1, 2$.



Domain: $x - y > 0 \rightarrow x > y$

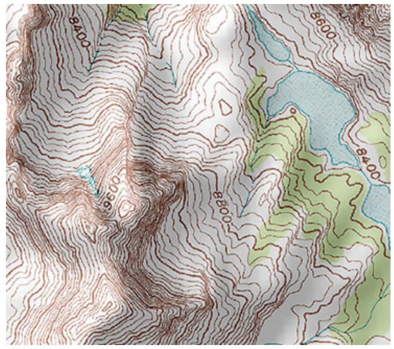
Range: $f = \mathbb{R}$

$$\ln(x-y) = c$$
$$x - y = e^c$$
$$y = x - e^c$$
$$y = x - 0.135 \qquad (c = -2)$$
$$y = x - 0.367 \qquad (c = -1)$$
$$y = x - 1 \qquad (c = 0)$$
$$y = x - 2.718 \qquad (c = 1)$$
$$y = x - 7.389 \qquad (c = 2)$$



Level curves:

Example 93. Topography



Exercise. Sketch the domain of

$$f(x, y) = \frac{4y + 2x}{x^2 + 2xy - 3}$$
$$g(x, y) = \ln(x) - \sqrt{y}$$

20 Limits

Suppose

$$\lim_{(x,y)\to(a,b)} f(x,y) = L_1 \quad and \quad \lim_{(x,y)\to(a,b)} g(x,y) = L_2$$

Then

$$\lim_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

$$\lim_{(x,y)\to(a,b)} [cf(x,y)] = c \lim_{(x,y)\to(a,b)} f(x,y)$$

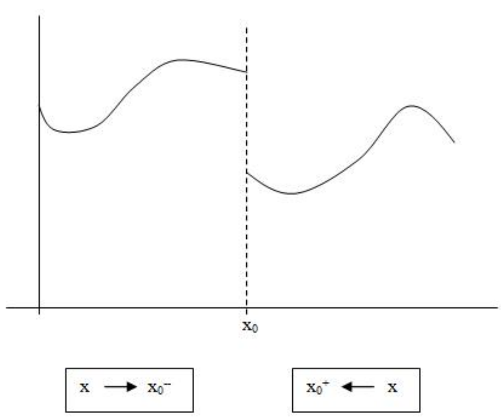
$$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \lim_{(x,y)\to(a,b)} g(x,y)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} \text{ if the denominator is not zero}$$

Example 94.

$$\lim_{(x,y)\to(0,0)} x^2+y^2 = 0^2+0^2 = 0$$
$$\lim_{(x,y)\to(4,-3)} x^2+y^2 = 4^2+(-3)^2 = 16+9 = 25$$
$$\lim_{(x,y)\to(2,0)} \frac{4y+2x}{x^2+2xy-3} = \frac{4(0)+2(2)}{2^2+(2)(2)(0)-3} = \frac{4}{4-3} = 4$$

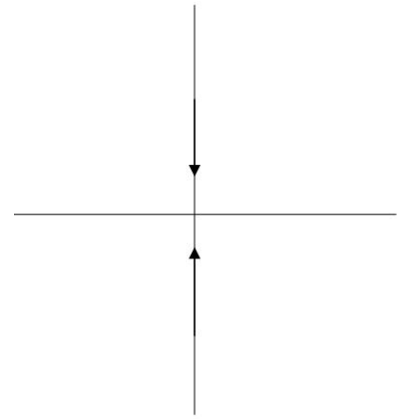
Recall: $f(x)$ is continuous at x_0 if $\lim_{x\to x_0^-} h(x) = \lim_{x\to x_0^+} h(x)$.
For one-variable functions, there are two paths: from the left and from the right. For multi-variable functions, there are many paths.



Example 95. Is $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous at (0,0)?
Path 1: Along the x-axis (so y=0)

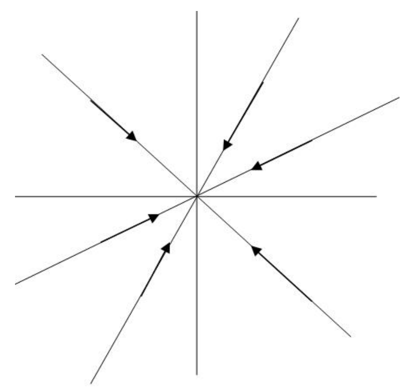
$$\lim_{(x,0)\to(0,0)} f = \lim_{x\to 0} \frac{x^2-0}{x^2+0} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Path 2: Along the y-axis (so x=0)



$$\lim_{(0,y)\to(0,0)} f = \lim_{y\to 0} \frac{0-y^2}{0+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

Therefore the limit does not exist and f(x,y) is not continuous at (0,0).
More generally, we could approach by paths $y = mx$.



$$\lim_{(x,mx)\to(0,0)} f = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2}$$

The value of the limit is different for different values of m. Thus, the limit does not exist.

Example 96. Show that $\lim_{(x,y)\to(0,0)} \frac{4xy}{xy+y^3}$ does not exist.
Straight lines:

$$\lim_{(x,mx)\to(0,0)} \frac{4xy}{xy+y^3} = \lim_{x\to 0} \frac{4mx^2}{mx^2+(mx)^3} = \lim_{x\to 0} \frac{4}{1+m^2x} = 4$$

So it looks like the limit exists.
But let's try the path $x = y^2$.

$$\lim_{(y^2,y)\to(0,0)} \frac{4xy}{xy+y^3} = \lim_{y\to 0} \frac{4y^3}{y^3+y^3} = \frac{4}{2} = 2 \neq 4$$

Therefore, the limit does not exist.

21 Partial Derivatives

For functions with two or more variables, we don't just have one derivative, but rather a derivative for each variable. Partial derivatives keep all other variables fixed and look at the rate of change of only the variable of interest

Example 97.

$$
\begin{aligned}
f(x,y) &= x^2 + 3xy + y^4 + 6 \\
\frac{\partial f}{\partial x} &= 2x + 3y \\
\frac{\partial f}{\partial y} &= 3x + 4y^3
\end{aligned}
$$

Example 98. Use the chain rule and product rule for $g(x,y) = x^2 e^{xy}$ to find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

$$
\frac{\partial g}{\partial x} = (2x + x^2 y)e^{xy} \qquad \frac{\partial g}{\partial y} = x^3 e^{xy}
$$

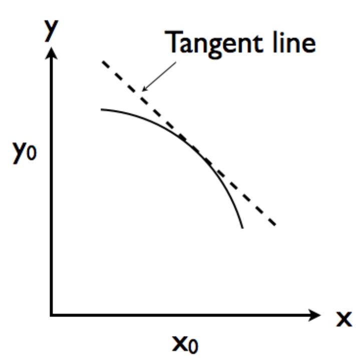
Example 99. $h(x,y) = \frac{\cos(xy)}{y^2+1}$. Find $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$.

$$
\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{-\sin(xy)y}{y^2+1} = \frac{-y\sin(xy)}{y^2+1} \\
\frac{\partial h}{\partial y} &= \frac{(y^2+1)(-\sin(xy)x) - \cos(xy)(2y)}{(y^2+1)^2} \\
&= \frac{-x(y^2+1)\sin(xy) - 2y\cos(xy)}{(y^2+1)^2}
\end{aligned}
$$

Example 100. $k(x,y,z) = e^{yz}(x^2 + z^3)$. Find all partial derivatives.

$$
\begin{aligned}
\frac{\partial k}{\partial x} &= 2xe^{yz} \\
\frac{\partial k}{\partial y} &= ze^{yz}(x^2 + z^3) \\
\frac{\partial k}{\partial z} &= ye^{yz}(x^2 + z^3) + 3z^2 e^{yz}
\end{aligned}
$$

Geometric Interpretation: for one-variable functions, $\frac{df}{dx}$ is the slope of the tangent line at x. The equation of the tangent line is $(y - y_0) = f'(x_0)(x - x_0)$.

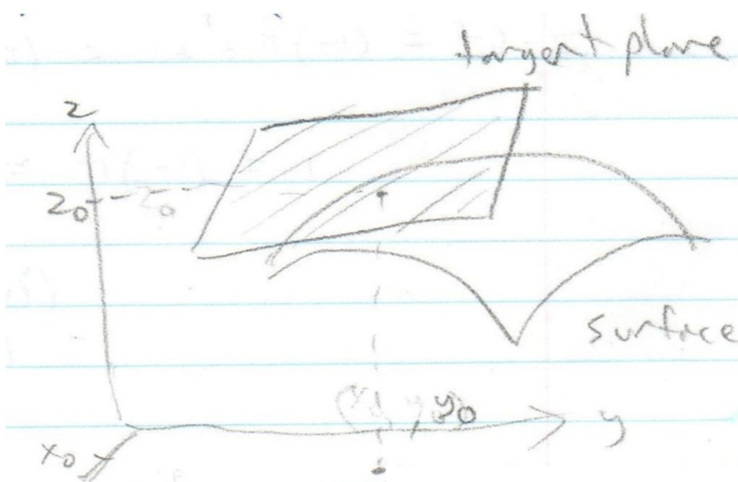


It's similar for two variables, but now instead of a tangent line, we have a tangent plane. If $z = f(x, y)$, then the tangent plane is

$$(z - z_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Note: The tangent plane might not exist if the limit does not exist. But then the tangent line might not exist either.

Example 101. Find the equation of the tangent plane to the surface $z = 4x^2 + y^2$ at the point (1,2,8).



$$\begin{aligned}\frac{\partial z}{\partial x} &= 8x \\ \frac{\partial z}{\partial y} &= 2y \\ z - 8 &= 8(1)(x - 1) + 2(2)(y - 2) \\ &= 8(x - 1) + 4(y - 2) \\ &= 8x - 8 + 4y - 8 \\ z &= 8x + 4y - 8\end{aligned}$$

Example 102. Find the equation for the tangent planes to $z = x^2 + \sin(xy)$ when

a) (x, y)=(1,0) b) (x, y)=(0,1) c)(x, y)=(-1,π)

Solution:

a)

$z(1,0) = 1^2 + \sin(0) = 1$

$$\frac{\partial z}{\partial x} = 2x + \cos(xy)y$$
$$\frac{\partial z}{\partial x}(1,0) = 2(1) + \cos(0)0 = 2$$
$$\frac{\partial z}{\partial y}(1,0) = \cos(xy)x$$
$$\frac{\partial z}{\partial y} = \cos(0)(1) = 1$$
$$z - 1 = 2(x-1) + 1(y-0)$$
$$z - 1 = 2x - 2 + y$$
$$z = 2x + y - 1$$

b)

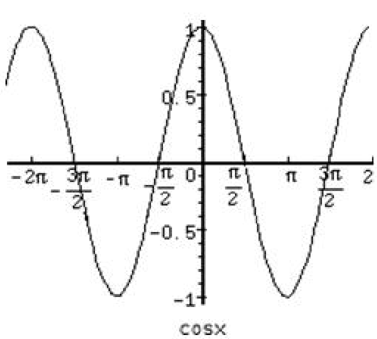
$z(0,1) = 0^2 + \sin(0) = 0$

$$\frac{\partial z}{\partial x} = 2(0) + 1\cos(0) = 1$$
$$\frac{\partial z}{\partial y} = (0)\cos(0) = 0$$
$$z - 0 = 1(x-0) + 0(y-1)$$
$$z = x$$

c)

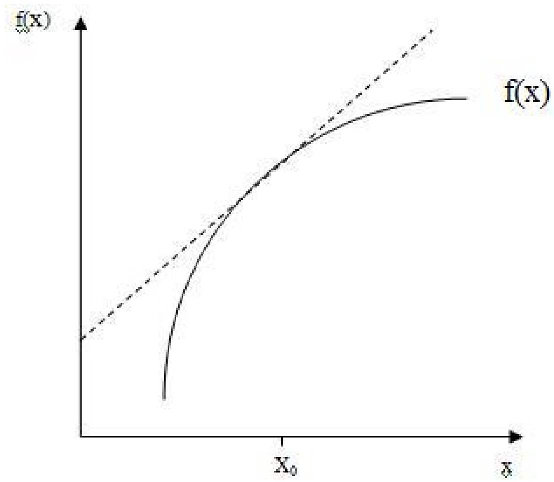
$z(-1,\pi) = (-1)^2 + \sin(-\pi) = 1$

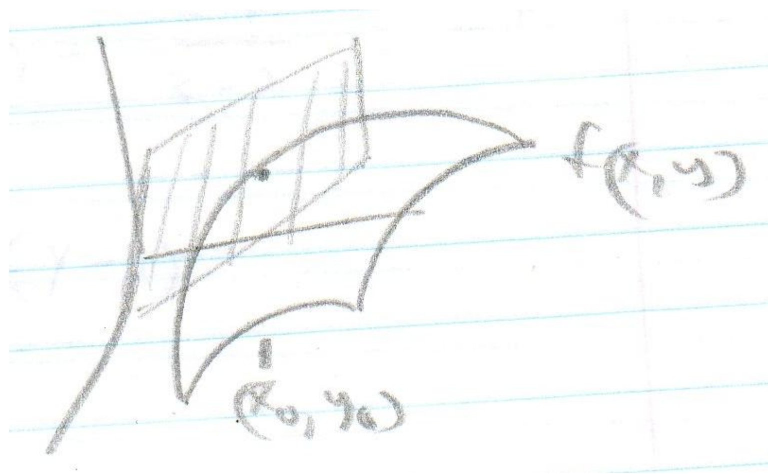
$$\frac{\partial z}{\partial x} = 2(-1) + \pi\cos(-\pi) = -2 + \pi(-1) = -2 - \pi$$
$$\frac{\partial z}{\partial y} = (-1)\cos(-\pi) = -1(-1) = 1$$
$$z - 1 = (-2-\pi)(x+1) + 1(y-\pi)$$
$$z - 1 = -(2+\pi)x - 2 - \pi + y - \pi$$
$$z = -(2+\pi)x + y - 1 - 2\pi$$



21.1 Linearisation

We can approximate difficult functions by their tangent plane at the point of interest. If we're close to this point then the tangent plane is close to the function.





Definition 21.1. *Suppose $f(x, y)$ is differentiable at (x_0, y_0). The* linearization of $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

The approximation $f(x, y) \approx L(x, y)$ is the standard linear appoximation of $f(x, y)$ at (x_0, y_0).

Example 103. Find the linear approximation of $f(x, y) = x^2y + 2xe^y$ at (2,0).

$$f(2, 0) = 0 + 4 = 4$$

$$\frac{\partial f}{\partial x} = 2xy + 2e^y$$
$$\frac{\partial f}{\partial x}(2, 0) = 0 + 2 = 2$$
$$\frac{\partial f}{\partial y} = x^2 + 2xe^y$$
$$\frac{\partial f}{\partial y}(2, 0) = 4 + 4 = 8$$

$$L(x, y) = 4 + 2(x - 2) + 8(y - 0) = 2x + 8y$$

Example 104. a) Find the linear approximation of $f(x, y) = \ln(x - 2y^2)$ at (3,1).
b) How close is this approximation to the original function at (3.05,0.95)?

a)

$$
\begin{aligned}
f(3,1) &= 0 \\
\frac{\partial f}{\partial x} &= \frac{1}{x-2y^2} \\
\frac{\partial f}{\partial x}(3,1) &= 1 \\
\frac{\partial f}{\partial y} &= \frac{-4y}{x-2y^2} \\
\frac{\partial f}{\partial y}(3,1) &= -4 \\
L(x,y) &= 0 + 1(x-3) - 4(y-1) \\
&= x - 3 - 4y + 4 \\
&= x - 4y + 1
\end{aligned}
$$

b)

$$
\begin{aligned}
f(3.05, 0.95) &= \ln(3.05 - 2(0.95)^2) = 0.2191 \\
L(3.05, 0.95) &= 3.05 - 4(0.95) + 1 = 0.25
\end{aligned}
$$

The error of approximation is $|0.25 - 0.2191| = 0.031$.
[The second midterm covers up to here.]