Midterm MATH 251, October 21, 2016

Justify all answers

Problem 1 [6 pt]
a. [**2**] Prove that the following two subsets are subspaces of $\mathcal{P}_3(\mathbb{R})$ (the space of polynomials of degree ≤ 3)

$$W_1 := \{p(x) \in \mathcal{P}_3(\mathbb{R}) : \quad 2p(0) + p(1) = 0\} \tag{1}$$
$$W_2 := \{p(x) \in \mathcal{P}_3(\mathbb{R}) : \quad p(2) = 0\} \tag{2}$$

b. [**2**] Find **bases** and **dimensions** of W_1, W_2
c. [**2**] Find a **base** and **dimension** of $W_1 \cap W_2$.

Problem 2 [4pt] Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three *linearly independent* vectors in the vector space V. Show that the three new vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ given below are also linearly independent

$$\mathbf{x} = \mathbf{u} + \mathbf{v} + \mathbf{w}\ , \quad \mathbf{y} = \mathbf{u} + \mathbf{v}\ , \quad \mathbf{z} = \mathbf{u} + \mathbf{w}. \tag{3}$$

Problem 3 [4 pt]
a.[**2**] State *in detail* the "dimension theorem" (a.k.a. "nullity + rank" theorem);
b.[**2**] Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Suppose that $\dim V = 2 + \dim W$. Is it possible for T to be one-to-one? If yes give an example, if not explain why.

Problem 4 [10 pt] Let $T : \mathcal{P}_3 \to \mathbb{R}^3$ be given by

$$T(p(x)) = \langle p(1), \quad p'(0), \quad p(1) - p'(0)\rangle. \tag{4}$$

a.[**3**] Verify that $\beta = \{x + 2,\ x - 3,\ x^2 + 1,\ x^3\}$ is a basis of $\mathcal{P}_3$
b.[**3**] Let $\gamma = \{\langle 1, 1, 0\rangle, \langle 1, -1, 0\rangle, \langle 0, 0, 1\rangle\}$ be a basis of $\mathbb{R}^3$ (you don't need to verify this). Compute $[T]_\beta^\gamma$.
c.[**2**] Find **a base and the dimension** of the range $\mathbf{R}(T)$
d.[**2**] Find **a base and the dimension** of the null-space $\mathbf{N}(T)$

Problem 5 [6 pt] Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $Q = \{\mathbf{u}_1, \mathbf{u}_2\}$ be two subsets of a vector space V consisting of mutually distinct vectors;

1. Is it true that $(\operatorname{Span} S) \cap (\operatorname{Span} Q) = \operatorname{Span}(S \cap Q)$? If yes, prove it, if not, explain why not or give a counterexample.
2. Is it true that $(\operatorname{Span} S) + (\operatorname{Span} Q) = \operatorname{Span}(S \cup Q)$? If yes, prove it, if not, explain why not or give a counterexample.
3. If $\dim(\operatorname{Span} S) = 2$ then is it true that S is linearly dependent? Prove or disprove.
4. If $\dim(\operatorname{Span} Q) = 2$ then is it true that Q is linearly dependent? Prove or disprove.

Problem 6 [Bonus 3 pt] Let $T : V \to W$ and $U : V \to W$ be two linear transformations between the indicated vector spaces V, W. Prove that $\mathbf{N}(U) \cap \mathbf{N}(T) \subseteq \mathbf{N}(U + T)$. Give an example where the inclusion is strict.

Solution to Problem 1 **a.** For W_1 if $p, q \in W_1$ then $2p(0) + p(1) = 0$ and $2q(0) + q(1) = 0$. Then, we need to see if the polynomial $h(x) = p(x) + aq(x)$ belongs to W_1, where here $a \in \mathbb{R}$ is an arbitrary scalar.

$$2h(0) + h(1) = 2\big(p(0) + aq(0)\big) + \big(p(1) + aq(1)\big) = \underbrace{2p(0) + p(1)}_{=0} + a\big(\underbrace{2q(0) + q(1)}_{=0}\big) = 0. \tag{5}$$

So $h(x) \in W_1$.

Similarly, if $p, q \in W_2$ then $p(2) = 0$ and $q(2) = 0$. Then $h(2) = p(2) + aq(2) = 0$.

So both W_1, W_2 are subspaces.

b. We write $p(x) = ax^3 + bx^2 + cx + d$; then

$$2p(0) + p(1) = a + b + c + 3d = 0. \tag{6}$$

Thus $d = -\frac{a+b+c}{3}$. The general polynomial of W_1 is

$$p(x) = ax^3 + bx^2 + cx - \frac{a+b+c}{3} = a\underbrace{\left(x^3 - \frac{1}{3}\right)}_{q_1} + b\underbrace{\left(x^2 - \frac{1}{3}\right)}_{q_2} + c\underbrace{\left(x - \frac{1}{3}\right)}_{q_3} \in \text{Span}\left\{q_1, q_2, q_3\right\} \tag{7}$$

The polynomials q_1, q_2, q_3 are independent because the LHS of (7) is zero iff $a = b = c = 0$. Thus they are a basis and the dimension is 3.

For W_2 we have

$$p(2) = 8a + 4b + 2c + d = 0 \quad \Rightarrow \quad d = -8a - 4b - 2c \quad \Rightarrow \tag{8}$$

$$p(x) = ax^3 + bx^2 + cx - 8a - 4b - 2c = a\underbrace{\left(x^3 - 8\right)}_{w_1} + b\underbrace{\left(x^2 - 4\right)}_{w_2} + c\underbrace{\left(x - 2\right)}_{w_3} \in \text{Span}\{w_1, w_2, w_3\} \tag{9}$$

By the same reason as above, they form a basis and the dimension is 3.

c. For $W_1 \cap W_2$ we have the system

$$\left\{\begin{array}{c} a + b + c + 3d = 0 \\ 8a + 4b + 2c + d = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} d = -\frac{a+b+c}{3} \\ 8a + 4b + 2c - \frac{a+b+c}{3} = \frac{23}{3}a + \frac{11}{3}b + \frac{5}{3}c = 0 \end{array}\right. \tag{10}$$

$$\Rightarrow \left\{\begin{array}{c} d = -\frac{a+b+c}{3} \\ c = -\frac{23}{5}a - \frac{11}{5}b \end{array}\right. \Rightarrow \left\{\begin{array}{c} d = -\frac{a+b}{3} + \frac{23}{15}a + \frac{11}{15}b = \frac{6}{5}a + \frac{2}{5}b \\ c = -\frac{23}{5}a - \frac{11}{5}b \end{array}\right. \tag{11}$$

Thus the polynomial in the intersection is of the form

$$p(x) = ax^3 + bx^2 + \left(-\frac{23}{5}a - \frac{11}{5}b\right)x + \left(\frac{6}{5}a + \frac{2}{5}b\right) = a\underbrace{\left(x^3 - \frac{23}{5}x + \frac{6}{5}\right)}_{s_1} + b\underbrace{\left(x^2 - \frac{11}{5}x + \frac{2}{5}\right)}_{s_2} \tag{12}$$

For the same reasons as above, s_1, s_2 are independent and form a basis of $W_1 \cap W_2$. The dimension is 2. ■

Solution to Problem 2

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = a\left(\mathbf{u} + \mathbf{v} + \mathbf{w}\right) + b\left(\mathbf{u} + \mathbf{v}\right) + c\left(\mathbf{u} + \mathbf{w}\right) = (a + b + c)\mathbf{u} + (a + b)\mathbf{v} + (a + c)\mathbf{w} = 0 \tag{13}$$

Since u, v, w are independent, we must have

$$\left\{\begin{array}{c} a + b + c = 0 \\ a + b = 0 \\ a + c = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} -a = 0 \\ a + b = 0 \\ a + c = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} a = 0 \\ b = 0 \\ c = 0 \end{array}\right. \tag{14}$$

Thus x, y, z must be independent because the only combination that gives the null vector is the trivial one. ■

Solution to Problem 3 a. Let $T : V \to W$ be a linear map between the vector spaces V, W. Suppose that V is finite-dimensional. Then

$$rank(T) + nullity(T) = \dim V. \tag{15}$$

b. The rank of T cannot exceed the dimension of W; so $rank(T) \leq \dim W = \dim V - 2$. Plugging into the dimension theorem we obtain

$$nullity(T) = \dim V - rank(T) \geq \dim V - \dim W = 2 \tag{16}$$

So, the nullity of T is at least 2 and hence the map cannot be 1-1. ■

Solution to Problem 4 a.

$$a(x+2) + b(x-3) + c(x^2+1) + dx^3 = 0 \quad \Leftrightarrow \quad dx^3 + cx^2 + (a+b)x + (2a - 3b + c) \equiv 0 \quad \Leftrightarrow \tag{17}$$

$$\left\{ \begin{array}{c} d = 0 \\ c = 0 \\ a + b = 0 \\ 2a + 3b + c = 0 \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{c} d = 0 \\ c = 0 \\ a = 0 \\ b = 0 \end{array} \right. \tag{18}$$

So they are independent, and they are $4 = \dim \mathcal{P}_3$. So they are a basis.

b. Let v_1, v_2, v_3 be the two vectors in γ.

$$\begin{aligned} T(x+2) &= \langle 3, 1, 2 \rangle = 2v_1 + 1v_2 + 2v_3 && (19) \\ T(x-3) &= \langle -2, 1, -3 \rangle = -\frac{1}{2}v_1 - \frac{3}{2}v_2 - 3v_3 && (20) \\ T(x^2+1) &= \langle 2, 0, 2 \rangle = v_1 + v_2 + 2v_3 && (21) \\ T(x^3) &= \langle 1, 0, 1 \rangle = \frac{1}{2}v_1 + \frac{1}{2}v_2 + 1v_3 && (22) \end{aligned}$$

$$[T]_\beta^\gamma = \left[\begin{array}{cccc} 2 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 2 & -3 & 2 & 1 \end{array} \right] \tag{23}$$

c. It should be rather apparent that the range is contained in the subspace $\langle a, b, a-b \rangle$ of dimension 2. So it can't be of dimension bigger than 2. At this point it suffices to select 2 linearly independent vectors in the image of the basis. We have

$$\mathbf{R}(T) = \text{Span}\left\{\langle 3, 1, 2 \rangle,\ \langle 1, 0, 1 \rangle\right\} \tag{24}$$

which are clearly independent. Thus $rank(T) = 2$.

d. We must have $p(1) = 0$ and $p'(0) = 0$; thus $p(x) = ax^3 + bx^2 + cx + d$ must have $a + b + c + d = 0$ and $c = 0$

$$\left\{ \begin{array}{c} d = -a - b \\ c = 0 \end{array} \right. \quad \Rightarrow \tag{25}$$

$$p(x) = ax^3 + bx^2 + -a - b = a\underbrace{(x^3 - 1)}_{p_1} + b\underbrace{(x^2 - 1)}_{p_2} \in \text{Span}\{p_1, p_2\} \tag{26}$$

The vectors p_1, p_2 are independent (as seen by an argument used before) and hence they form a basis of $N(T)$; the dimension is 2.

To be noted that $2 + 2 = 4$ as predicted by the dimension theorem. ■

Solution to Problem 5 **1.** It is false, for example $S = \{[1,1],[1,2],[2,2]\}$ and $Q = \{[3,3],[5,5]\}$. Then $S \cap Q = \emptyset$ and hence $Span(S \cap Q) = \{[0,0]\}$ but it is clear that $SpanQ \subset SpanS$ and hence the intersection is $SpanS$ which is not trivial

2. It is true. if $v \in Span\,S + Span\,Q$ then

$$v = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) + (b_1\mathbf{u}_1 + b_1\mathbf{u}_1) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + b_1\mathbf{u}_1 + b_1\mathbf{u}_1 \in Span(S \cup Q). \tag{27}$$

Viceversa if $v \in Span(S \cup Q)$ then

$$v = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + b_1\mathbf{u}_1 + b_1\mathbf{u}_1 = \underbrace{(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3)}_{\in Span\,S} + \underbrace{(b_1\mathbf{u}_1 + b_1\mathbf{u}_1)}_{\in Span\,Q} \in Span\,S + Span\,Q \tag{28}$$

3. True; if they were independent then the dimension of $SpanS$ would be 3.

4. False. If Q were dependent then u_1 would be proportional to u_2 and the dimension would be 1 or 0. ■

Solution to Problem 6 If $v \in \mathbf{N}(T) \cap \mathbf{N}(U)$ then $T(v) = 0_W = U(v)$. But then $(U+T)(v) = U(v) + T(v) = 0_W + 0_W = 0_W$ and hence $v \in \mathbf{N}(U+T)$.

To give an example where the inclusion is strict; Let $T = Id_{\mathbb{R}^2}$ and $U = -T$. Then $\mathbf{N}(T) = \mathbf{N}(U) = \{0\}$. However $\mathbf{N}(T+U) = \mathbf{N}(T_0) = \mathbb{R}^2$. ■