MAT 2384C-Winter 2016-Assignment #2
To be submitted in class, Wednesday February 10

Last Name Solutions

First Name ____________________

Student Number ____________________

- **Please print this assignment and write solutions in the provided space.**
- If needed, use the back of the pages or the additional pages provided to write your solutions.
- There are 6 questions in this assignment.
- You must answer all the questions.
- Some of the questions may not be marked.
- Your solution should be written clearly and in a readable manner.

Question 1. [6 points] Solve the following IVP.

$$\underbrace{\left(ye^{\sin(x)}\cos(x) - y^3 + 2xy\right)}_{M} dx + \underbrace{\left(2e^{\sin(x)} - 4y^2(x+1) + 2x^2\right)}_{N} dy, \quad y(0) = 2.$$

$\frac{\partial M}{\partial y} = e^{\sin x}\cos x - 3y^2 + 2x$; $\frac{\partial N}{\partial x} = 2\cos x\, e^{\sin x} - 4y^2 + 4x$; Not exact

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -e^{\sin x}\cos x + y^2 - 2x \Rightarrow \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \dfrac{-(e^{\sin x}\cos x - y^2 + 2x)}{y(e^{\sin x}\cos x - y^2 + 2x)} = -\frac{1}{y} = g(y)$

So an integrating factor exists and is given by

$\mu(y) = e^{-\int g(y)\,dy} = e^{\int 1/y\,dy} = e^{\ln y} = y$. Multiply the ODE with y:

$$\underbrace{\left(y^2 e^{\sin x}\cos x - y^4 + 2xy^2\right)}_{M^*} dx + \underbrace{\left(2ye^{\sin x} - 4y^3(x+1) + 2x^2 y\right)}_{N^*} dy = 0$$

$\frac{\partial M^*}{\partial y} = 2ye^{\sin x}\cos x - 4y^3 + 4xy$; $\frac{\partial N^*}{\partial x} = 2y\cos x\, e^{\sin x} - 4y^3 + 4xy$; Now exact.

We look for a function $F(x,y)$ satisfying : $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$

$\frac{\partial F}{\partial y} = 2ye^{\sin x} - 4y^3(x+1) + 2x^2 y \Rightarrow F(x,y) = y^2 e^{\sin x} - y^4(x+1) + x^2y^2 + h(x)$. So

$\frac{\partial F}{\partial x} = y^2\cos(x)e^{\sin x} - y^4 + 2xy^2 + h'(x)$. Comparing with $\frac{\partial F}{\partial x} = M^*$, we get :

$h'(x) = 0 \Rightarrow h(x) = k = $ Constant. So, $F(x,y) = y^2 e^{\sin x} - y^4(x+1) + x^2y^2 + k$.

The general solution is $F(x,y) = $ Constant $\Leftrightarrow y^2 e^{\sin x} - y^4(x+1) + x^2y^2 = C$.

Now, $y(0) = 2 \Rightarrow 4e^0 - 16(0+1) + 0 = C \Rightarrow C = -12$.

The IVP solution is $\boxed{y^2 e^{\sin x} - y^4(x+1) + x^2y^2 = -12}$

Question 2. **[6 points]** Use the Secant Method to estimate $\sqrt[3]{13}$ to 5 decimal places. Start with $x_0 = 2$, $x_1 = 2.5$.

Let $x = \sqrt[3]{13}$, then $x^3 = 13 \Leftrightarrow x^3 - 13 = 0$. Let $f(x) = x^3 - 13$.

By the Secant Method, we have $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 2.5 - (2.5^3 - 13)\frac{2.5 - 2}{(2.5^3 - 13) - (2^3 - 13)} = 2.32787$$

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)} = 2.32787 - (2.32787^3 - 13)\frac{2.32787 - 2.5}{(2.32787^3 - 13) - (2.5^3 - 13)} = 2.34990$$

$$x_4 = x_3 - f(x_3)\frac{x_3 - x_2}{f(x_3) - f(x_2)} = 2.34990 - (2.34990^3 - 13)\frac{2.34990 - 2.32787}{(2.3499^3 - 13) - (2.32787^3 - 13)} = 2.35135$$

$$x_5 = x_4 - f(x_4)\frac{x_4 - x_3}{f(x_4) - f(x_3)} = 2.35135 - (2.35135^3 - 13)\frac{2.35135 - 2.34990}{(2.35135^3 - 13) - (2.34990^3 - 13)} = 2.35133$$

$$x_6 = x_5 - f(x_5)\frac{x_5 - x_4}{f(x_5) - f(x_4)} = 2.35133 - (2.35133^3 - 13)\frac{2.35133 - 2.35135}{(2.35133^3 - 13) - (2.35135^3 - 13)} = 2.35133$$

We conclude that $\sqrt[3]{13} \simeq 2.35133$ correct to 5 decimal place.

Question 3. [6 points] Solve each of the following IVP's.

1. $y' + 2y = 5e^{-4x}, \quad y(0) = -3.$
2. $y' + \frac{2}{x} y = x, \quad y(4) = 3.$

1) Linear first order ODE with $f(x) = 2$, $r(x) = 5e^{-4x}$. The general solution is

$$y = \frac{\int e^{\int f(x)dx} \cdot r(x)\,dx + C}{e^{\int f(x)dx}} = \frac{\int e^{\int 2dx}\, 5e^{-4x}\,dx + C}{e^{\int 2dx}} = \frac{5\int e^{2x} e^{-4x}\,dx + C}{e^{2x}}$$

$$= \frac{5\int e^{-2x}\,dx + C}{e^{2x}} = \frac{-5/2\, e^{-2x} + C}{e^{2x}} = -\frac{5}{2} e^{-4x} + Ce^{-2x}.$$

Now, $y(0) = -3 \Rightarrow -\frac{5}{2} + C = -3 \Rightarrow C = -1/2$. The solution to the PVI is

$$\boxed{y = -\frac{1}{2}\left(5e^{-4x} + e^{-2x}\right)}$$

2) Linear first order ODE with $f(x) = \frac{2}{x}$, $r(x) = x$. The general solution is

$$y = \frac{\int e^{\int f(x)dx}\, r(x)\,dx + C}{e^{\int f(x)dx}} = \frac{\int e^{\int 2/x\,dx}\, x\,dx + C}{e^{\int 2/x\,dx}} = \frac{\int e^{2\ln x}\, x\,dx + C}{e^{2\ln x}} = \frac{\int e^{\ln(x^2)}\, x\,dx + C}{e^{\ln(x^2)}}$$

$$= \frac{\int x^3\,dx + C}{x^2} = \frac{\frac{1}{4}x^4 + C}{x^2} = \frac{1}{4}x^2 + \frac{C}{x^2}$$

Now $y(4) = 3 \Rightarrow 3 = \frac{1}{4}(16) + \frac{C}{16} \Rightarrow \frac{C}{16} = -1 \Rightarrow C = -16.$

The solution to the PVI is $\boxed{y = \frac{1}{4}x^2 - \frac{16}{x^2}}$

Question 4. [6 points] Solve each of the following IVP's.

1. $y' + y\tan x = y^2, \quad y(0) = 1/2, \; -\frac{\pi}{2} < x < \pi/2$
2. $y' + \frac{3}{x}y = \frac{x}{y}, \quad y(1) = 1/2 \; , \; x > 0$

1) Bernouilli equation with $a = 2$. Let $U = y^{1-a} = y^{-1} \Rightarrow U' = -y^{-2}y' \Rightarrow$ $U' = -y^{-2}(y^2 - y\tan x) = -1 + y^{-1}\tan x = -1 + U(\tan x) \Rightarrow U' - (\tan x)U = -1$: linear first order ODE with $f(x) = -\tan x$, $r(x) = -1$. The general solution is given by $u = \frac{\int e^{\int -\tan x\,dx}(-1)\,dx + C}{e^{\int(-\tan x)dx}}$. Note that $\int(-\tan x)\,dx =$

$\int \frac{-\sin x}{\cos x}\,dx = \ln|\cos x| = \ln(\cos x)$ since $\cos x > 0$ for $x \in]-\pi/2, \pi/2[$

so $e^{\int -\tan x\,dx} = e^{\ln(\cos x)} = \cos x$ and the general solution is:

$$u = \frac{-\int \cos x\,dx + C}{\cos x} = \frac{-\sin x + C}{\cos x} = -\tan x + \frac{C}{\cos x}$$

Now $y(0) = 1/2 \Rightarrow U(0) = \frac{1}{y(0)} = 2 \Rightarrow 2 = -\tan 0 + \frac{c}{\cos 0} \Rightarrow C = 2$

$$u = -\tan x + \frac{2}{\cos x} = \frac{2 - \sin x}{\cos x} \Rightarrow y = \frac{1}{u} = \boxed{\frac{\cos x}{2 - \sin x}}$$

2) Bernouilli equation with $a = -1$. Let $u = y^{1-a} = y^2 \Rightarrow U' = 2yy' =$ $2y\left(\frac{x}{y} - \frac{3}{x}y\right) = 2x - \frac{6}{x}y^2 = 2x - \frac{6}{x}U \Rightarrow U' + \frac{6}{x}U = 2x$: linear first order ODE with $f(x) = \frac{6}{x}$, $r(x) = 2x$. The general solution is:

$$u = \frac{\int e^{\int 6/x\,dx}\,2x\,dx + C}{e^{\int 6/x\,dx}} = \frac{2\int e^{6\ln x}\,x\,dx + C}{e^{6\ln x}} = \frac{2\int x^7\,dx + C}{x^6} = \frac{\frac{1}{4}x^8 + C}{x^6}$$

$= \frac{1}{4}x^2 + \frac{C}{x^6}$. Now $y(1) = 1/2 \Rightarrow u(1) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow$

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$\frac{1}{4} = \frac{1}{4}(1)^2 + \frac{c}{(1)^c} \Rightarrow c = 0$. So $u = \frac{1}{4}x^2$.

We conclude that $y^2 = \frac{1}{4}x^2 \Rightarrow y = \sqrt{\frac{1}{4}x^2} = \frac{1}{2}x$ since

$y(1) = \frac{1}{2} > 0$.

Question 5. [6 points] Consider the points (x_i, f_i), $i = 0, 1, 2$ where $f_i = f(x_i)$ for a certain unknown function f:

$$(0.5, 0.62054), (0.9, 0.58971), (1.2, 0.39694), (1.7, -0.16799).$$

It is also known that $0.00258 \le |f^{(4)}(x)| \le 16.08507$ on the interval $[0.5, 1.7]$.

1. Find the interpolation polynomial $p_3(x)$ of degree 3 via Lagrange. Round the coefficients to 5 decimal places. Interpolate a value for $f(1)$. Give an upper and a lower estimates on the absolute error for the estimation of $f(1)$.
2. Find the interpolation polynomial $p_3(x)$ of degree 3 via Newton. Round the coefficients to 5 decimal places. Interpolate a value for de $f(1.5)$. Give an upper and a lower estimates on the absolute error for the estimation of $f(1.5)$.

1) Via Lagrange: $P_3(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-0.9)(x-1.2)(x-1.7)}{(0.5-0.9)(0.5-1.2)(0.5-1.7)} = -2.97619x^3+11.30952x^2-13.83929x+5.46429$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0.5)(x-1.2)(x-1.7)}{(0.9-0.5)(0.9-1.2)(0.9-1.7)} = 10.41667x^3-35.41667x^2+36.35417x-10.62500$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0.5)(x-0.9)(x-1.7)}{(1.2-0.5)(1.2-0.9)(1.2-1.7)} = -9.52381x^3+29.52381x^2-26.95238x+7.28571$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0.5)(x-0.9)(x-1.2)}{(1.7-0.5)(1.7-0.9)(1.7-1.2)} = 2.08333x^3-5.41667x^2+4.43750x-1.12500$$

So the Lagrange interpolation polynomial is given by:

$$P_3(x) = (-2.97619x^3+11.30952x^2-13.83929x+5.46429)0.62054 +$$
$$(10.41667x^3-35.41667x^2+36.35417x-10.62500)(0.58971)+$$
$$(-9.52381x^3+29.52381x^2-26.95238x+7.28571)(0.39694)+$$
$$(2.08333x^3-5.41667x^2+4.43750x-1.12500)(-0.16799) \Rightarrow$$

$$\boxed{P_3(x) = 0.16561x^3-1.23842x^2+1.40665x+0.20612}$$

$f(1) \simeq P_3(1) = 0.16561(1)^3 - 1.23842(1)^2 + 1.40665(1) + 0.20612 = \boxed{0.53995}$

The absolute value of the error in the approximation $f(1) \simeq P_3(1)$ is given by $|f(1) - P_3(1)| = |(1-0.5)(1-0.9)(1-1.2)(1-1.7)| \frac{|f^{(4)}(t)|}{4!} = 0.00029\,|f^{(4)}(t)|$

As $0.00258 \leq |f^{(4)}(t)| \leq 16.08507$ on $[0.5, 1.7]$,

- The minimal absolute error is $0.00029(0.00258) = 0.0000007482$
- The maximal " " is $0.00029(16.08567) = 0.00046468$

2) Via Newton:

$P_3(x) = f_0 + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$

$f[x_0,x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{0.58971 - 0.62054}{0.9 - 0.5} = -0.07708$

$f[x_1,x_2] = \frac{f_2 - f_1}{x_2 - x_1} = \frac{0.39694 - 0.58971}{1.2 - 0.9} = -0.64257$

$f[x_2,x_3] = \frac{f_3 - f_2}{x_3 - x_2} = \frac{-0.16799 - 0.39694}{1.7 - 1.2} = -1.12986$

$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \frac{-0.64257 - (-0.07708)}{1.2 - 0.5} = -0.80785$

$f[x_1,x_2,x_3] = \frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} = \frac{-1.12986 - (-0.64257)}{1.7 - 0.9} = -0.60912$

$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0} = \frac{-0.60912 - (-0.80785)}{1.7 - 0.5} = 0.16561$

The Newton interpolation polynomial is

$P_3(x) = 0.62054 - 0.07708(x-0.5) - 0.80785(x-0.5)(x-0.9) +$
$0.16561(x-0.5)(x-0.9)(x-1.2) \Rightarrow$

$\boxed{P_3(x) = 0.16561x^3 - 1.23842x^2 + 1.40665x + 0.20612}$

$f(1.5) \simeq P_3(1.5) = \boxed{0.08857}$

- Minimal absolute error is $\frac{|(1.5-0.5)(1.5-0.9)(1.5-1.2)(1.5-1.7)|}{4!}(0.00258)$

$= 0.0000387$ and the Maximal absolute error is 0.0241276

Question 6. [6 points] Solve each of the following IVP's.

1. $y'' + 2\sqrt{3}y' + 3y = 0, \quad y(0) = \sqrt{3}, \quad y'(0) = 0.$
2. $y'' - 6y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = 0.$
3. $y'' - 8y' + 15y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

1) The characteristic equation is $\lambda^2 + 2\sqrt{3}\lambda + 3 = 0 \Leftrightarrow (\lambda + \sqrt{3})^2 = 0 \Leftrightarrow$ $\lambda = -\sqrt{3}$ is a double root. A basis for solutions is $y_1 = e^{-\sqrt{3}x}$, $y_2 = xe^{-\sqrt{3}x}$ and the general solution is $y = c_1 e^{-\sqrt{3}x} + c_2 x e^{-\sqrt{3}x}$

$y' = -\sqrt{3}c_1 e^{-\sqrt{3}x} + c_2 e^{-\sqrt{3}x} - \sqrt{3}c_2 x e^{-\sqrt{3}x}$

$y(0) = \sqrt{3} \Rightarrow c_1 = \sqrt{3}$

$y'(0) = 0 \Rightarrow -\sqrt{3}c_1 + c_2 = 0 \Rightarrow c_2 = \sqrt{3}c_1 = 3$

The solution to the IVP is $\boxed{y = \sqrt{3}e^{-\sqrt{3}x} + 3xe^{-\sqrt{3}x}}$

2) The characteristic equation is $\lambda^2 - 6\lambda + 13 = 0$. The solutions are:

$\lambda = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$. A basis for solutions is given by

$y_1 = e^{3x}\cos(2x)$, $y_2 = e^{3x}\sin(2x)$. The general solution is $y = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$.

$y' = 3c_1 e^{3x}\cos(2x) - 2c_1 e^{3x}\sin(2x) + 3c_2 e^{3x}\sin(2x) + 2c_2 e^{3x}\cos(2x)$

$y(0) = 2 \Rightarrow c_1 = 2$

$y'(0) = 0 \Rightarrow 3c_1 + 2c_2 = 0 \Rightarrow c_2 = -3$

The solution to the IVP is $\boxed{y = 2e^{3x}\cos(2x) - 3e^{3x}\sin(2x)}$

3) The characteristic equation is $\lambda^2 - 8\lambda + 15 = 0$. The roots are $\lambda_1 = 3$, $\lambda_2 = 5$.
A basis for solutions is given by $y_1 = e^{3x}$, $y_2 = e^{5x}$ and the general solution is $y = c_1 e^{3x} + c_2 e^{5x}$. We have $y' = 3c_1 e^{3x} + 5c_2 e^{5x}$.

$y(0) = 2 \Rightarrow c_1 + c_2 = 2$

$y'(0) = -1 \Rightarrow 3c_1 + 5c_2 = -1$

$\Rightarrow c_1 = 11/2, \quad c_2 = -7/2$

The solution to the IVP is $\boxed{y = \frac{11}{2}e^{3x} - \frac{7}{2}e^{5x}}$