MAT 2384C-Winter 2016-Assignment #1
To be submitted in class, Wednesday 27 January

Last Name Solutions

First Name

Student Number

- **Please print this assignment and write solutions in the provided space.**
- If needed, use the back of the pages or the additional pages provided to write your solutions.
- There are 4 questions in this assignment.
- You must answer all the questions.
- Some of the questions may not be marked.
- Your solution should be written clearly and in a readable manner.

Question 1. [6 points] Solve the following IVP's.

1. $(1+y^4)\,y' - y\cos x = 0, \quad y(0) = 1$

2. $2x^2 + 3y^2 + xyy' = 0, \quad x > 0, \quad y(1) = 2$

1) $(1+y^4)\dfrac{dy}{dx} - y\cos x = 0 \Rightarrow \dfrac{1+y^4}{y}\,dy = \cos x\,dx$: separable ODE

$\int \dfrac{1+y^4}{y}\,dy = \int \cos x\,dx \Rightarrow \int\left(\dfrac{1}{y} + y^3\right)dy = \sin x + C \Rightarrow \ln y + \dfrac{1}{4}y^4 = \sin x + C$.

$y(0) = 1 \Rightarrow \ln 1 + \dfrac{1}{4}(1)^4 = \sin 0 + C \Rightarrow C = \dfrac{1}{4}$, so

$\ln y + \dfrac{1}{4}y^4 = \sin x + \dfrac{1}{4} \Rightarrow \boxed{4\ln y + y^4 = 4\sin x + 1}$: implicit solution.

2) $2x^2 + 3y^2 + xyy' = 0 \Leftrightarrow \underbrace{(2x^2+3y^2)}_{M}\,dx + \underbrace{xy}_{N}\,dy = 0$.

Note that $M(\lambda x, \lambda y) = 2\lambda^2 x^2 + 3\lambda^2 y^2 = \lambda^2(2x^2+3y^2) = \lambda^2 M(x,y)$
$N(\lambda x, \lambda y) = (\lambda x)(\lambda y) = \lambda^2 xy = \lambda^2 N(x,y)$

the ODE is homogeneous of degree 2. Let $u = \dfrac{y}{x} \Rightarrow y = xu \Rightarrow$

$\dfrac{dy}{dx} = u + x\dfrac{du}{dx} \Rightarrow dy = u\,dx + x\,du$. The ODE become:

$(2x^2 + 3x^2u^2)\,dx + x^2u(u\,dx + x\,du) = 0 \Rightarrow (2+3u^2)\,dx + (u^2\,dx + xu\,du) = 0$

$\Rightarrow (2+4u^2)\,dx + xu\,du = 0 \Rightarrow \dfrac{u}{2+4u^2}\,du = -\dfrac{1}{x}\,dx$; separable ODE.

$\int \dfrac{u}{2+4u^2}\,du = \int -\dfrac{1}{x}\,dx = -\ln x$. Let $t = 2+4u^2 \Rightarrow \dfrac{dt}{du} = 8u \Rightarrow du = \dfrac{dt}{8u}$

$\int \dfrac{u}{2+4u^2}\,du = \int \dfrac{u}{t}\,\dfrac{dt}{8u} = \dfrac{1}{8}\int \dfrac{1}{t}\,dt = \dfrac{1}{8}\ln t = \dfrac{1}{8}\ln(2+4u^2)$. So:

$\dfrac{1}{8}\ln(2+4u^2) = -\ln x + C$; $y(1) = 2 \Rightarrow u(1) = \dfrac{2}{1} = 2$, so: $\dfrac{1}{8}\ln(18) = C$

$\Rightarrow \dfrac{1}{8}\ln(2+4u^2) = -\ln x + \dfrac{1}{8}\ln(18) \Rightarrow \ln(2+4u^2) = -8\ln x + \ln(18)$

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$$\Rightarrow \ln(2+4u^2) = \ln\left(\frac{1}{x^8}\right) + \ln(18) = \ln\left(\frac{18}{x^8}\right) \Rightarrow 2+4u^2 = \frac{18}{x^8} \Rightarrow$$

$$u^2 = \frac{1}{4}\left[\frac{18}{x^8} - 2\right] = \frac{1}{2}\left(\frac{9-x^8}{x^8}\right) = \frac{9-x^8}{2x^8}.$$

$u = \sqrt{\dfrac{9-x^8}{2x^8}} = \dfrac{\sqrt{9-x^8}}{\sqrt{2}\,x^4}$ (since $u(1) = 2 > 0$). We conclude that

$$y = xu = \boxed{\frac{\sqrt{9-x^8}}{\sqrt{2}\,x^3}}$$

Question 2. **[8 points]** Solve the following IVP's.

1. $(2xy^3 + \cos y)dx + (-x\sin y + 3x^2y^2)dy = 0, \quad y\left(\frac{\pi}{2}\right) = 2$
2. $\left(x\ln(y^4) + 4\ln y - y^2\right)dx + \left(-2y + \frac{4x}{y}\right)dy = 0, \quad y(0) = 1$

1) $M(x,y) = 2xy^3 + \cos y$; $N(x,y) = -x\sin y + 3x^2y^2$

$\frac{\partial M}{\partial y} = 6xy^2 - \sin y$; $\frac{\partial N}{\partial x} = -\sin y + 6xy^2$: Exact ODE. We look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

$\frac{\partial F}{\partial x} = 2xy^3 + \cos y \Rightarrow F = \int (2xy^3 + \cos y)\,dx = x^2y^3 + x\cos y + h(y)$. So

$\frac{\partial F}{\partial y} = 3x^2y^2 - x\sin y + h'(y)$. Comparing with $\frac{\partial F}{\partial y} = N$, we find that

$h'(y) = 0 \Rightarrow h(y) = k$. So $F(x,y) = x^2y^3 + x\cos y + k$. The general solution of the ODE is $x^2y^3 + x\cos y = C$. Since $y(\pi/2) = 2$, we get:

$\frac{\pi^2}{4}(4) + \frac{\pi}{2}\cos(\pi/2) = C \Rightarrow C = \pi^2$. The solution of the IVP is :

$$\boxed{x^2y^3 + x\cos y = \pi^2}$$

2) $M(x,y) = x\ln y^4 + 4\ln y - y^2$; $N(x,y) = -2y + \frac{4x}{y}$

$\frac{\partial M}{\partial y} = \frac{4xy^3}{y^4} + \frac{4}{y} - 2y = \frac{4x}{y} + \frac{4}{y} - 2y$; $\frac{\partial N}{\partial x} = \frac{4}{y}$: ODE Not Exact.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{4x}{y} - 2y \Rightarrow \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 1 = f(x)$. An integration factor exists and given by $\mu(x) = e^{\int f(x)dx} = e^{\int 1 dx} = e^x$. We multiply the ODE by the factor e^x;

$$\underbrace{\left(xe^x\ln(y^4) + 4e^x\ln y - y^2e^x\right)}_{M^*}dx + \underbrace{\left(-2e^xy + \frac{4xe^x}{y}\right)}_{N^*}dy = 0$$

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$\frac{\partial M^*}{\partial y} = \frac{4xe^x}{y} + \frac{4e^x}{y} - 2ye^x$; $\frac{\partial N^*}{\partial x} = -2e^x y + \frac{4e^x}{y} + \frac{4xe^x}{y}$: ODE Exact.

We look at a function $F(x,y)$ satisfying $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$.

$\frac{\partial F}{\partial y} = -2e^x y + \frac{4xe^x}{y} \Rightarrow F(x,y) = -e^x y^2 + 4xe^x \ln y + h(x)$. So

$\frac{\partial F}{\partial x} = -e^x y^2 + 4e^x \ln y + 4xe^x \ln y + h'(x)$. Comparing with $\frac{\partial F}{\partial x} = M^*$, we get that $h'(x) = 0$ (remember that $xe^x \ln(y^4) = 4xe^x \ln y$). So $h(x) = K =$ Constant. We conclude that the general solution is $F(x,y) =$ Constant

$\Rightarrow -e^x y^2 + 4xe^x \ln y = C$.

$y(0) = 1 \Rightarrow -1 = C$. The solution to the PVI is :

$-e^x y^2 + 4xe^x \ln y = -1 \Rightarrow \boxed{e^x y^2 - 4xe^x \ln y = 1}$

Question 3. **[8 points]** Solve the following IVP's.

1. $(\cos x - 2x\sin x - 2y\sin x)\,dx + \cos x\,dy = 0, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y(0) = 1$
2. $(y + 2xy^3 + 2xy^4e^y)dx + (-3x + x^2y^4e^y - x^2y^2)dy = 0, \quad y(1) = 1$.

1) $M = \cos x - 2x\sin x - 2y\sin x$; $N = \cos x$

$\frac{\partial M}{\partial y} = -2\sin x$; $\frac{\partial N}{\partial x} = -\sin x$: Not Exact. Note that $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\sin x$

$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{\sin x}{\cos x} = -\tan x = f(x)$. An integrating factor exists and it is given by $\mu(x) = e^{\int f(x)dx} = e^{-\int \frac{\sin x}{\cos x}dx} = e^{\ln(\cos x)} = \cos x$ (note that $\cos x > 0$ for $-\pi/2 < x < \pi/2$). We multiply the ODE by $\cos x$:

$$\underbrace{(\cos^2 x - 2x\sin x\cos x - 2y\sin x\cos x)}_{M^*}dx + \underbrace{\cos^2 x}_{N^*}\,dy = 0$$

$\frac{\partial M^*}{\partial y} = -2\sin x\cos x$; $\frac{\partial N^*}{\partial x} = -2\cos x\sin x$: Exact. We look for a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M^*$ and $\frac{\partial F}{\partial y} = N^*$.

$\frac{\partial F}{\partial y} = N^* = \cos^2 x \Rightarrow F(x,y) = y\cos^2 x + h(x) \Rightarrow \frac{\partial F}{\partial x} = -2y\cos x\sin x + h'(x)$.

Comparing with $\frac{\partial F}{\partial x} = M^*$, we find that $h'(x) = \cos^2 x - 2x\sin x\cos x \Rightarrow$

$$h(x) = \int \cos^2 x\,dx - \int 2x\sin x\cos x\,dx = \int\left(\frac{1+\cos 2x}{2}\right)dx - \int x\sin(2x)\,dx$$

(using the trig identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin 2x = 2\sin x\cos x$).

$$\int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\left[\int 1\,dx + \int \cos(2x)dx\right] = \frac{1}{2}\left[x + \frac{\sin(2x)}{2}\right] = \frac{x}{2} + \frac{1}{4}\sin(2x).$$

For the integral $\int x\sin(2x)dx$, we use integration by parts:

$u = x$, $v' = \sin(2x) \Rightarrow u' = 1$, $v = -\frac{\cos(2x)}{2}$; so:

$$\int x\sin(2x)\,dx = -\frac{x}{2}\cos(2x) - \int -\frac{\cos(2x)}{2}\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x)$$

So $h(x) = \frac{x}{2} + \frac{1}{4}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) = \frac{x}{2} + \frac{x}{2}\cos(2x)$.

Therefore $F(x,y) = y\cos^2 x + \frac{x}{2} + \frac{x}{2}\cos(2x)$. The general solution of the ODE is given by

$$y\cos^2 x + \frac{x}{2} + \frac{x}{2}\cos(2x) = C.$$

$y(0) = 1 \Rightarrow C = 1$. The solution of the IVP is

$$\boxed{y\cos^2 x + \frac{x}{2} + \frac{x}{2}\cos(2x) = 1.}$$

2) $M = y + 2xy^3 + 2xy^4e^y$; $N = -3x + x^2y^4e^y - x^2y^2$

$\frac{\partial M}{\partial y} = 1 + 6xy^2 + 8xy^3e^y + 2xy^4e^y$; $\frac{\partial N}{\partial x} = -3 + 2xy^4e^y - 2xy^2$: Not Exact.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4 + 8xy^2 + 8xy^3e^y = 4(1 + 2xy^2 + 2xy^3e^y) \Rightarrow$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4(1+2xy^2+2xy^3e^y)}{y(1+2xy^2+2xy^3e^y)} = \frac{4}{y} = g(y).$$ An integrating factor exists and is given by:

$$\mu(y) = e^{-\int g(y)dy} = e^{-\int \frac{4}{y}dy} = e^{-4\ln y} = e^{\ln(y^{-4})} = y^{-4}.$$ We multiply the ODE by y^{-4}:

$$\underbrace{(y^{-3} + 2xy^{-1} + 2xe^y)}_{M^*}dx + \underbrace{(-3xy^{-4} + x^2e^y - x^2y^{-2})}_{N^*}dy = 0$$

$\frac{\partial M^*}{\partial y} = -3y^{-4} - 2xy^{-2} + 2xe^y$; $\frac{\partial N^*}{\partial x} = -3y^{-4} + 2xe^y - 2xy^{-2}$; Exact. We look for a function $F(x,y)$ satisfying : $\frac{\partial F}{\partial x} = M^*$, $\frac{\partial F}{\partial y} = N^*$:

$\frac{\partial F}{\partial x} = y^{-3} + 2xy^{-1} + 2xe^y \Rightarrow F(x,y) = xy^{-3} + x^2y^{-1} + x^2e^y + h(y) \Rightarrow$

$\frac{\partial F}{\partial y} = -3xy^{-4} - x^2y^{-2} + x^2e^y + h'(y)$. Comparing with N^* yields :

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$h'(y) = 0 \Rightarrow h(y) = K =$ Constant . So $F(x,y) = xy^{-3} + x^2y^{-1} + x^2e^y + K$.

The general solution of the ODE is $xy^{-3} + x^2y^{-1} + x^2e^y = C$.

$y(1) = 1 \Rightarrow C = e + 2$. The solution of the IVP is:

$xy^{-3} + x^2y^{-1} + x^2e^y = e + 2 \iff \boxed{x + x^2y^2 + x^2y^3e^y = (e+2)y^3}$

Question 4. [7 points]

1. Use the **fixed point iteration** method to find the root of $f(x) = x^3+7x-6$ in the interval $[0,1]$ to 5 decimal places. Make sure that the conditions for convergence of the iteration sequence are satisfied. Start with $x_0 = 0.75$.

2. Use **Newton's Method** to find $\sqrt[3]{9}$ to 6 decimal places. Start with $x_0 = 2$.

3. Use **Newton's Method** to find the positive solution of $e^x = 2\cos x$ to 6 decimal places. Start with $x_0 = 1$.

1) $f(x)$ is clearly continuous on $[0,1]$ (in fact, in $\mathbb{R}$). Moreover, $f(0) = -6 < 0$ and $f(1) = 2 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for some $c \in [0,1]$.

$f(x) = 0 \iff x^3+7x-6=0 \iff x = \frac{6-x^3}{7}$. Let $g(x) = \frac{6-x^3}{7}$. Then g is clearly continuous on $[0,1]$. Moreover, $g'(x) = -\frac{3x^2}{7}$ is also continuous in $[0,1]$ and $|g'(x)| = \frac{3}{7}x^2 \leq \frac{3}{7}(1) = \frac{3}{7}$ for any $x \in [0,1]$. So $|g'(x)| \leq \frac{3}{7} < 1$ on $[0,1]$. We conclude that the iteration sequence $x_{n+1} = g(x_n)$ converges for any initial value x_0.

$x_0 = 0.75 \Rightarrow x_1 = g(x_0) = \frac{6-0.75^3}{7} = 0.79688$

$x_2 = g(x_1) = \frac{6-0.79688^3}{7} = 0.78485$; $x_3 = g(x_2) = \frac{6-0.78485^3}{7} = 0.78808$

$x_4 = g(x_3) = \frac{6-0.78808^3}{7} = 0.78722$; $x_5 = g(x_4) = \frac{6-0.78722^3}{7} = 0.78745$

$x_6 = g(x_5) = \frac{6-0.78745^3}{7} = 0.78739$; $x_7 = g(x_6) = \frac{6-0.78739^3}{7} = 0.78740$

$x_8 = g(x_7) = \frac{6-0.787405^3}{7} = 0.78740$

We conclude that the solution to $x^3+7x-6=0$ in $[0,1]$ is $x \simeq 0.78740$ correct to 5 decimal places.

2) Let $x = \sqrt[3]{9}$, then $x^3 = 9$ or $x^3-9=0$. Let $f(x) = x^3-9$. Clearly, f is continuous and $f'(x) = 3x^2$. By the Newton's method we have $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3-9}{3x_n^2}$.

$$x_0 = 2 \Rightarrow x_1 = x_0 - \frac{x_0^3-9}{3x_0^2} = 2.083333$$

$$x_2 = x_1 - \frac{x_1^3-9}{3x_1^2} = 2.080089; \quad x_3 = x_2 - \frac{x_2^3-9}{3x_2^2} = 2.080084$$

$$x_4 = x_3 - \frac{x_3^3-9}{3x_3^2} = 2.080084.$$ So $\sqrt[3]{9} = 2.080084$ correct to 6 decimal place

3) $e^x = 2\cos x \Leftrightarrow e^x - 2\cos x = 0$. The function $f(x) = e^x - 2\cos x$ is continuous as well as its derivative $f'(x) = e^x + 2\sin x$.

$$x_0 = 1 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{e^1 - 2\cos 1}{e^1 + 2\sin 1} = 0.627904$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.627904 - \frac{e^{0.627904} - 2\cos(0.627904)}{e^{0.627904} + 2\sin(0.627904)} = 0.544207$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.544207 - \frac{e^{0.544207} - 2\cos(0.544207)}{e^{0.544207} + 2\sin(0.544207)} = 0.539797$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.539797 - \frac{e^{0.539797} - 2\cos(0.539797)}{e^{0.539797} + 2\sin(0.539797)} = 0.539785$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.539785 - \frac{e^{0.539785} - 2\cos(0.539785)}{e^{0.539785} + 2\sin(0.539785)} = 0.539785$$

We conclude that the positive solution to the equation $e^x = 2\cos x$ is $x = 0.539785$ correct to 6 decimal place.