Midterm Test 203 2017

1. (12)

   (a) solve for $x : \log_{10} x + \log_{10}(x-3) = 1$

   (b) Sketch the graph $f(x) = |\ln(x-2)+1|$ from $g(x) = \ln x$. Also determine the $x - \mathit{intercepts}$ of $f(x)$.

   (c) If $f(x) = \sqrt[3]{e^x - e}$, calculate $f^{-1}(x)$ & and determine the range of both functions.

2. (6) For the graph $y = \dfrac{2x+3}{\sqrt{x^2-2x-3}}$ calculate horizontal and vertical asymptotes

3. (8) Calculate the limits, or explain why the limit does not exist:

   (a) $\lim\limits_{x\to\infty^-} \left(\sqrt{x^2-2x} - x\right)$

   (b) $\lim\limits_{x\to-2} \dfrac{x^2+x-6}{|x-2|}$

4. (4) Let $f(x) = \dfrac{1+\sin x}{1-\sin x}$. Calculate $f''(x)$ and the exact value of $f''(0)$.

5. (16) Calculate the derivatives of:

   (a) $f(x) = x^2\sqrt{x^5}\left(x - x^{-1}\right)$

   (b) $f(x) = \left(e^3 + x^3\right)3^x$

   (c) $f(x) = \dfrac{x^2 \tan x}{1+\cos 2x}$

   (d) $f(x) = \sin(x\sin x + \sin(x + \sin x))$

6. (4) From definition calculate the derivative of $f(x) = \dfrac{3x}{x+2}$

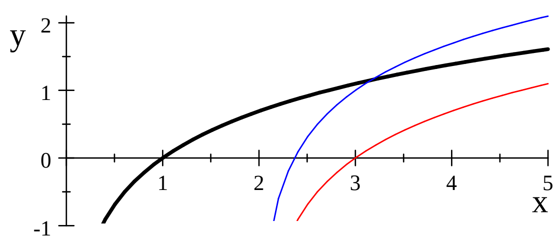
Bonus (3) Consider: $f(x) = \begin{cases} x+a & \text{if } x \le 1 \\ ax^2+b & \text{if } x > 1 \end{cases}$. Determine the values a and b so $f(x)$ is differentiable everywhere, or explain why it is impossible.

Midterm Test 203 2017
(Solutions)

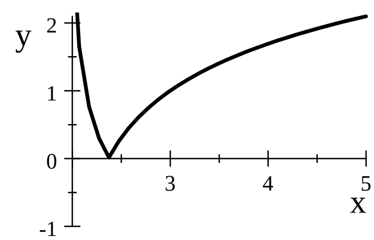
1. (12)

   (a) solve for $x : \log_{10} x + \log_{10}(x-3) = 1$
   $\log_{10} x(x-3) = \log_{10} 10 \to x(x-3) = 10, \to x = \{5, -2\}$.
   Since $x = 5 : \log_{10} 5 + \log_{10} 2 = 1$ it works & $x = -2 : \log_{10}(-2)$ ...*reject*. $\to x = 5$

   (b) Sketch the graph $f(x) = |\ln(x-2) + 1|$ from $g(x) = \ln x$. Also determine the $x - intercepts$ of $f(x)$.



$y = \ln x, \ln(x-2)$ blue, $\ln(x-2)+1$ red $\to$



$y = |\ln(x-2)+1|$

   No $x - intercepts$.

   (c) If $f(x) = \sqrt[3]{e^x - e}$, calculate $f^{-1}(x)$ & and determine the range of both functions.
   $x = \sqrt[3]{e^y - e} \to x^3 = e^y - e \to x^3 + e = e^y \to y = f^{-1}(x) = \ln\left(x^3 + e\right)$.
   Range of $f(x)$ is $(-\sqrt[3]{e}, \infty)$, range of $f^{-1}(x) = (-\infty, \infty)$.

2. (6) For the graph $y = \dfrac{2x+3}{\sqrt{x^2 - 2x - 3}}$ calculate horizontal and vertical asymptotes

   (a) $\lim\limits_{x\to\infty} \dfrac{2x+3}{\sqrt{x^2-2x-3}} = \lim\limits_{x\to\infty} \dfrac{2 + \frac{3}{x}}{\sqrt{1 - \frac{2}{x} - \frac{3}{x^2}}} = 2 \to y = 2$ is a horizontal asymptote

   & $\lim\limits_{x\to-\infty} \dfrac{2x+3}{\sqrt{x^2-2x-3}} \overset{as\ \sqrt{x^2}=-x}{=} \lim\limits_{x\to-\infty} \dfrac{-\left(2 + \frac{3}{x}\right)}{\sqrt{1 - \frac{2}{x} - \frac{3}{x^2}}} = -2$ is also a horizontal asymptote

   (b) As $\dfrac{2x+3}{\sqrt{x^2-2x-3}} = \dfrac{2x+3}{\sqrt{(x+1)(x-3)}} \to \mathbf{D} = (-\infty, -1) \cup (3, \infty) \to$

$\lim\limits_{x\to -1^-} \dfrac{2x+3}{\sqrt{(x+1)(x-3)}} = \infty$ and $\lim\limits_{x\to 3^+} \dfrac{2x+3}{\sqrt{(x+1)(x-3)}} = \infty$ the vertical asymptotes are $x=-1$ & $x=3$.

3. (8) Calculate the limits, or explain why the limit does not exist:

   (a) $\lim\limits_{x\to\infty^-} \left(\sqrt{x^2-2x}-x\right)$

   $= \lim\limits_{x\to\infty^-} \dfrac{\left(\sqrt{x^2-2x}-x\right)\left(\sqrt{x^2-2x}+x\right)}{\left(\sqrt{x^2-2x}+x\right)} =$

   $\lim\limits_{x\to\infty^-} \dfrac{x^2-2x-x^2}{\sqrt{x^2-2x}+x} = \lim\limits_{x\to\infty^-} \dfrac{-2x}{\sqrt{x^2-2x}+x} = \lim\limits_{x\to\infty^-} \dfrac{-2}{\sqrt{1-\dfrac{2}{x}}+1} =$

   -1

   (b) $\lim\limits_{x\to -2} \dfrac{x^2+x-6}{|x-2|}$

   As $\lim\limits_{x\to -2^-} \dfrac{x^2+x-6}{|x-2|} = \lim\limits_{x\to -2^-} \dfrac{(x+3)(x-2)}{-(x-2)} = -1$ and $\lim\limits_{x\to -2^+} \dfrac{(x+3)(x-2)}{(x-2)} =$ $1 \neq -1$, the limit does not exist.

4. (4) Let $f(x) = \dfrac{1+\sin x}{1-\sin x}$. Calculate $f''(x)$ and the exact value of $f''(0)$.

   Simplify : $f'(x) = \dfrac{\cos x(1-\sin x)+\cos x(1+\sin x)}{(1-\sin x)^2} = \dfrac{2\cos x}{(1-\sin x)^2} \to$

   $f''(x) = \dfrac{-2\sin x(1-\sin x)^2 + 4\cos x(1-\sin x)\cos x}{(1-\sin x)^4} =$

   $\dfrac{-2\sin x(1-\sin x)+4\cos x\cos x}{(1-\sin x)^3} \to f''(0) = 4.$

5. (16) Calculate the derivatives of:

   (a) $f(x) = x^2\sqrt{x^5}\left(x-x^{-1}\right)$

   $= x^3\sqrt{x^5} - x\sqrt{x^5} = x^{11/2} - x^{3/5} \to f'(x) = \dfrac{11}{2}x^{\frac{9}{2}} - \dfrac{3}{5}x^{-\frac{2}{5}}$

   (b) $f(x) = \left(e^3+x^3\right)3^x$

   $\to f'(x) = 3x^2 3^x + \left(e^3+x^3\right)3^x \ln 3 = 3^x\left(x^3\ln 3 + e^3\ln 3 + 3x^2\right)$

   (c) $f(x) = \dfrac{x^2\tan x}{1+\cos 2x}$

   $\to f'(x) = \dfrac{\left(2x\tan x + x^2\sec^2 x\right)(1+\cos 2x) + 2x^2\tan x\sin 2x}{(1+\cos 2x)^2}$

   (d) $f(x) = \sin\left(x\sin x + \sin(x+\sin x)\right)$

   $\to f'(x) = \cos\left(x\sin x + \sin(x+\sin x)\right)\left(\sin x + x\cos x + (1+\cos x)\cos(x+\sin x)\right)$

6. (4) From definition calculate the derivative of $f(x) = \dfrac{3x}{x+2}$

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{3(x+h)}{(x+h)+2} - \dfrac{3x}{x+2}}{h} = \lim_{h\to 0} \frac{3(x+h)(x+2) - 3x(x+h+2)}{h(x+2)(x+h+2)} =$$

$$\lim_{h\to 0} \frac{6\not{h}}{\not{h}(x+2)(x+h+2)} = \lim_{h\to 0} \frac{6}{(x+2)(x+h+2)} = \frac{6}{(x+2)^2}$$

Bonus (3) Consider: $f(x) = \begin{cases} x+a & \text{if } x \le 1 \\ ax^2+b & \text{if } x > 1 \end{cases}$. Determine the values a and b so $f(x)$ is differentiable everywhere, or explain why it is impossible:

(a) To make it continuous: $a = a + b \to b = 0$.

(b) To make it differentiable: $1 = 2a \to a = \dfrac{1}{2}$

(c) Therefore: $f(x) = \begin{cases} x + \dfrac{1}{2} & \text{if } x \le 1 \\ \dfrac{x^2}{2} & \text{if } x > 1 \end{cases}$ with the graph ($x \le 1$ in red & $x > 1$ in blue):