University of Ottawa
MAT 1330 3X Midterm Exam
June 18, 2013. Duration: 80 minutes.

Instructor: Catalin Rada

Family Name:______________________

First Name:______________________

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good Luck!

Student number:____________________, Total marks: __________ out of 30

Problem	1	2	3	4
Marks				

Question 1. [8 points] The concentration of a drug in the body of a patient is reduced by 30% per day. The daily dose of this drug is d. The DTDS modeling the concentration x_t of the drug in the body on day t is

$$x_{t+1} = 0.7x_t + d.$$

(a) [1 point] The updating function of the DTDS is $f(x) =$ $0.7x + d$

(b) [1 point] The equilibrium point of the DTDS is $x^* =$ $\frac{d}{0.3}$

(c) [1 point] Assume the daily dose is $d = 5$. Give the solution formula for the DTDS with general initial condition x_0:

$x_t =$ $(0.7)^t\left(x_0 - \frac{5}{0.3}\right) + \frac{5}{0.3}$

(d) [1 point] For a patient with an initial concentration of $x_0 = 3$ and a daily dose of $d = 5$, what is the concentration on day 3?

11.97

9. [**4 points**] Match each function with the corresponding series expansion.

$\dfrac{1}{1+x^2}$ •	• $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$
$\cos(2x)$ •	• $\displaystyle\sum_{n=1}^{\infty} n^2 x^{n-1}$
$\displaystyle\int_0^x \frac{1}{1-t}\,dt$ •	• $\displaystyle\sum_{n=0}^{\infty}\frac{(-4)^n x^{2n}}{(2n)!}$
$\displaystyle\frac{d}{dx}\left(\sum_{n=0}^{\infty} n x^n\right)$ •	• $\displaystyle\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}$

10. [**4 points**] Find the directional derivative of the function $f(x, y) = xe^{xy}$ at the point $(1, 1)$ in the direction of the vector $\vec{v} = \langle 3, 4\rangle = 3\hat{\imath} + 4\hat{\jmath}$.

A) e B) $6e/5$ C) $7e/5$ D) $8e/5$ E) $9e/5$ F) $2e$

b) $x^* = f(x^*) \Rightarrow x^* = 0.7x^* + d$

$\Rightarrow x^*(1-0.7) = d \Rightarrow x^* = \frac{d}{0.3}$

(c) so: $d = 5$, hence $x_{t+1} = \underset{a}{0.7}\, x_t + \underset{b}{5}$

From General Formula:

$$x_t = a^t(x_0 - x^*) + x^* =$$

$$= (0.7)^t\left(x_0 - \frac{5}{0.3}\right) + \frac{5}{0.3}$$

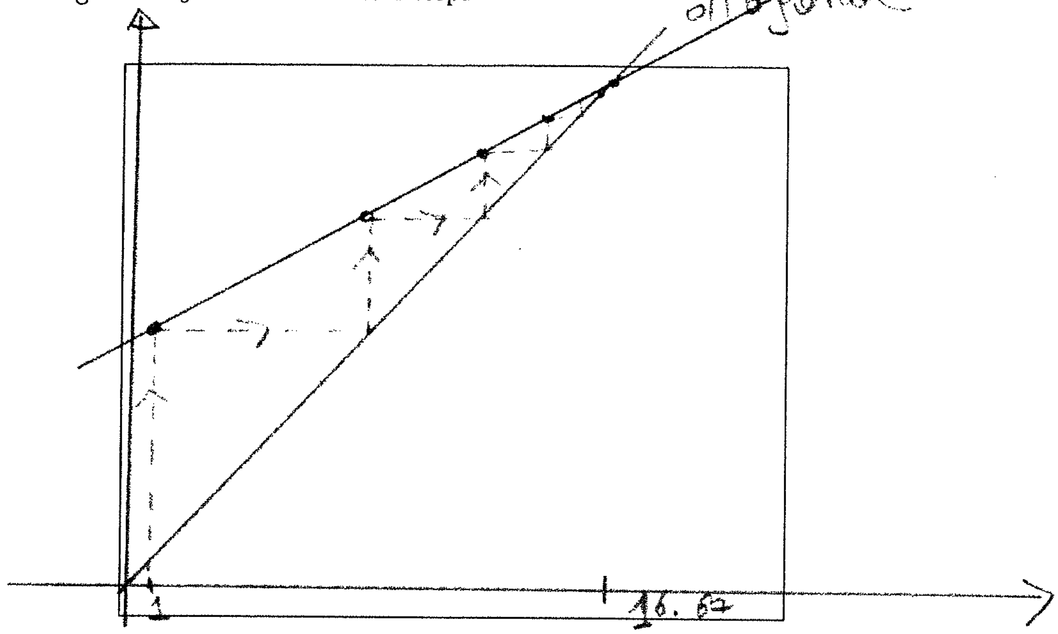
d) $x_3 = (0.7)^3\left(3 - \frac{5}{0.3}\right) + \frac{5}{0.3}$

$= \boxed{11.97}$

11. [**10 points**] Let $\mathcal{R}$ denote the region of the plane which is bounded by the curve $y = x^2$ and the horizontal line $y = 4$. Set $\mathcal{S}$ to be the solid of revolution obtained by rotating the region $\mathcal{R}$ about the vertical line $x = 3$.

Calculate the volume of $\mathcal{S}$ by the disc/washer method or the method of cylindrical shells (your choice). **Sketch the region $\mathcal{R}$, the section of the solid $\mathcal{S}$ which intersects the xy-plane, and an element of volume (disc/washer or cylinder) with its dimensions.**

(e) [2 points] Graph the updating function for $d = 5$ and draw the cobweb diagram of the DTDS, starting from $x_0 = 1$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable? STABLE

(g) [1 point] Suppose that the doctors recommend a concentration of 10 in the long run. How do they have to choose the daily dose d to obtain this value? $d =$ 3

$$10 = \lim_{t \to \infty} x_t = \lim_{t \to \infty} (0.7)^t \left(x_0 - \frac{d}{0.3}\right) + \frac{d}{0.3} = \frac{d}{0.3}$$

So: $d = (10) \times (0.3) = 3$

7. [**4 points**] Use the linear approximation of the function

$$f(x,y) = \sqrt{2 + 2x^2 - y^4}$$

at the point $(2,1)$ to approximate $f(2.05, 1.01)$.

A) 2.932 B) 2.96 C) 3 D) 3.06 E) 3.078 F) 3.1

8. [**4 points**] The radius r and height h of a cylinder vary as functions of time t. Set $S = 2\pi rh + 2\pi r^2$, the function giving its surface area. Given that at time $t = 2$ we have

$$r = 10 \text{ cm}, \quad h = 30 \text{ cm}, \quad \frac{dr}{dt} = 2 \text{ cm/s} \quad \text{and} \quad \frac{dh}{dt} = 1 \text{ cm/s},$$

determine the rate of change $\frac{dS}{dt}$ of the surface area at that instant.

A) 180π B) 200π C) 220π D) 240π E) 260π F) 280π

Question 2. [9 points] In each of the following cases, find the derivative of the function f with respect to variable x.

(a) $f(x) = \frac{10}{\sqrt{2}} e^{-(2x-10)^3}$

$$f'(x) = \frac{10}{\sqrt{2}} e^{-(2x-10)^3} \cdot \left[-(2x-10)^3\right]'$$

$$= \frac{10}{\sqrt{2}} e^{-(2x-10)^3} (-1) \cdot 3 (2x-10)^2 \cdot 2$$

(b) $f(x) = \ln\left(\frac{1}{x^2+x+2013}\right)$ $= \ln 1 - \ln(x^2+x+2013)$

$$f'(x) = -\frac{2x+1}{x^2+x+2013}$$

(c) $f(x) = e^{-x}\cos(-x) + \tan(2000x)$

$$f'(x) = e^{-x}(-1) \cdot \cos(-x) + e^{-x}(-\sin(-x))(-1) + \frac{1}{\cos^2(2000x)} \cdot 2000$$

Question 3. [5 points] (a) Use the definition of the derivative (first principles) to calculate the derivative of the function

$$f(x) = \sqrt{x^2 + x + 1}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2 + (x+h) + 1} - \sqrt{x^2 + x + 1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2 + (x+h)+1} - \sqrt{x^2+x+1}}{h\left[\sqrt{(x+h)^2 + (x+h)+1} + \sqrt{x^2+x+1}\right]} \cdot \frac{\sqrt{(x+h)^2 + (x+h)+1} + \sqrt{x^2+x+1}}{1}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 + (x+h) + 1 - x^2 - x - 1}{h\left[\sqrt{(x+h)^2 + (x+h)+1} + \sqrt{x^2+x+1}\right]} =$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + h}{h\left[\sqrt{(x+h)^2 + (x+h)+1} + \sqrt{x^2+x+1}\right]}$$

$$= \lim_{h\to 0} \frac{2x + h + 1}{\sqrt{(x+h)^2 + (x+h)+1} + \sqrt{x^2+x+1}} = \frac{2x + 0 + 1}{2\sqrt{x^2+x+1}}$$

(b) Check your result, using the differentiation rules from class.

By Chain Rule, Power rule:

$$f'(x) = \frac{1}{2}(x^2+x+1)^{-\frac{1}{2}} \cdot (2x+1)$$

Question 4. [8 points] Consider the function $f(x) = \frac{2x-6}{x-2}$.

(a) [1 point] Find the domain of f.

$$x \neq 2$$

(b) [1 point] Find the limits of f as x approaches $\pm\infty$

$$\lim_{x\to\infty} \frac{2x-6}{x-2} = \lim_{x\to 0} \frac{2-\frac{6}{x}}{1-\frac{2}{x}} = \frac{2-0}{1-0} = 2$$

$$\lim_{x\to-\infty} \frac{2x-6}{x-2} = \cdots\cdots = 2$$

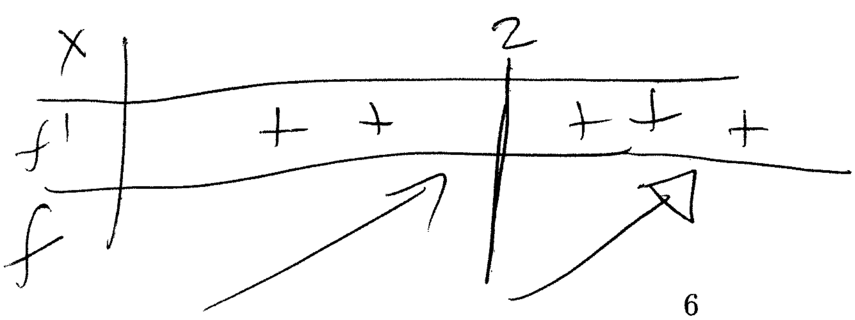
(c) [1 point] Are there points where f is not continuous? If yes, find the left and right limit in each case.

$$\lim_{x\to 2^-} \frac{2x-6}{x-2} = \frac{-2}{0^-} = \infty \qquad x<2$$

$$\lim_{x\to 2^+} \frac{2x-6}{x-2} = \frac{-2}{0^+} = -\infty \qquad x>2$$

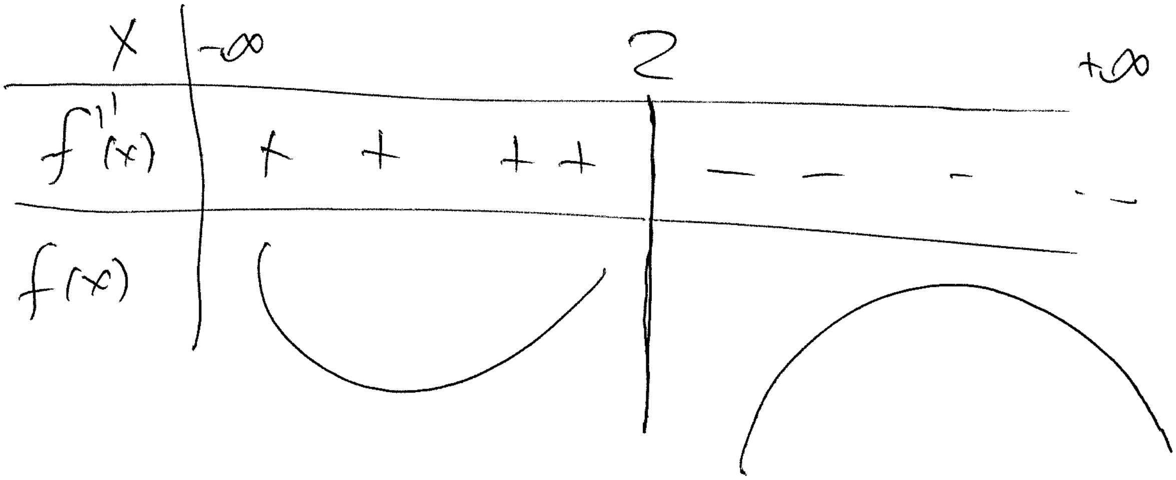
(d) [2 point] Find the intervals where f is increasing and decreasing. Are there critical points?

$$f'(x) = \frac{2(x-2)-(2x-6)\cdot 1}{(x-2)^2} = \frac{2x-4-2x+6}{(x-2)^2} = \frac{2}{(x-2)^2}$$



$$f''(x) = 2(-2)(x-2)^{-3}$$
$$= (-4) / (x-2)^3$$

(e) [2 point] Find the intervals where f is concave up or concave down.



(f) [1 point] Draw the graph of f.