1. (1 point) Suppose that a patient receives a daily dose of 50mg/L of a certain drug such that 55% of it is eliminated from the body each day. If on a certain Monday, the concentration of the drug measured in their body (shortly after the daily dose) is 42 mg/L, which of the following Discrete-Time Dynamical Systems describe the dynamics of the concentration x_t of the drug in the body (in mg/L, t days after that Monday)?

A. $x_{t+1} = .45x_t \cancel{+42}$, with $x_0 = 50$

B. $x_{t+1} = \cancel{.55x_t} + 42$, with $x_0 = 50$

C. $x_{t+1} = \cancel{.50x_t} + 45$, with $x_0 = 42$

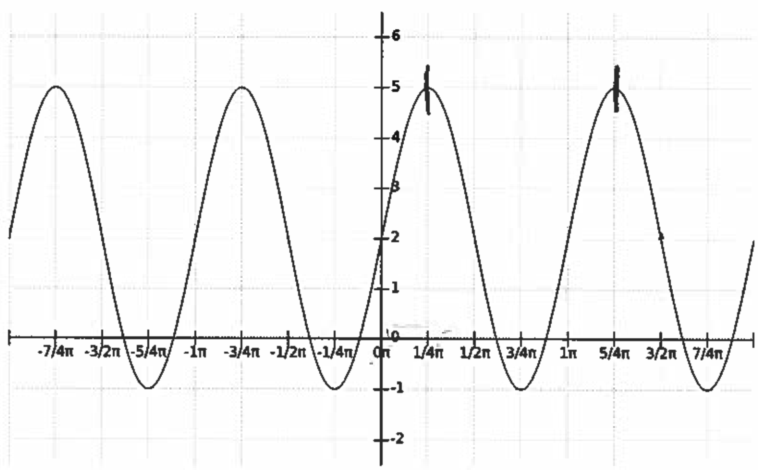
D. $x_{t+1} = .45x_t + 50$, with $x_0 = 42$

E. $x_{t+1} = \cancel{.55x_t} + 42$, with $x_0 = 45$

F. $x_{t+1} = \cancel{.42x_t} + 50$, with $x_0 = 45$

Your answer: D ✓

2. (1 point) The following is the graph of a function $y = f(x)$.



Which of the following is a formula for $f(x)$?

A. $f(x) = 3 + 2\cos\left(2(x - \frac{\pi}{2})\right)$

B. $f(x) = 2 + 3\cos\left(2(x - \frac{\pi}{4})\right)$

C. $f(x) = 1 + 3\cos\left(4(x - \frac{\pi}{2})\right)$

D. $f(x) = 2 + 3\cos\left(4(x - \frac{\pi}{2})\right)$

E. $f(x) = 2 + \cos\left(x - \frac{\pi}{2}\right)$

F. $f(x) = 3 + 2\cos\left(x + \frac{\pi}{4}\right)$

Your answer: B ✓

3. (1 point) Find the set of all solutions to the inequality

$$\frac{6}{x} - 8 < 4.$$

A. $(-\infty, \frac{1}{2})$

B. $(0, \frac{1}{2})$

C. $(\frac{1}{2}, \infty)$

D. $(0, \infty)$

E. $(-\infty, 0) \cup (\frac{1}{2}, \infty)$

F. $(-\infty, 0)$

Your answer: C

×

$\frac{6}{x} - 8 < 4$

$\frac{6}{x} < 4+8$

$\frac{6}{x} < 12$

Case 1

$\frac{6}{+x} < 12$

$6 < 12x$

$\frac{6}{12} < x$

$\frac{1}{2} < x$

or

$\boxed{x > \frac{1}{2}}$

Case 2

$\frac{6}{-x} < 12$

$6 > -12x$

$\frac{6}{-12} < x$

$x > -\frac{1}{2}$

$6 \cdot -\frac{1}{2} < 12$

$6 \cdot -2 < 12$

$-12 < 12$

$\frac{6}{x \frac{1}{4}} < 12$

$6 \times \frac{4}{1} < 12$

$(-\infty, \frac{1}{2}$

$12 < 12$

$6 \cdot 3 < 12$

$18 < 12$

4. (1 point) Find all x for which the following equality holds:

$$|6 - 2x^2| = 4.$$

A. $x = \pm 1$ or $x = \pm\sqrt{5}$

B. $x = 1$

C. $x = \pm 1$

D. $x = 0$

E. $x = \pm\sqrt{3}$

F. $x = 2 \pm \sqrt{3}$

Your answer: A

✓

Case 1

$6 - 2x^2 = 4$

$-2x^2 = -2$

$x^2 = 1$

$\boxed{x = \pm 1}$

Case 2

$-(6-2x^2) = 4$

$6 - 2x^2 = -4$

$-2x^2 = -10$

$x^2 = 5$

$\boxed{x = \pm\sqrt{5}}$

5. (2 points) Find all solutions x of the following equation. Show your work.

$$\ln(x+1) + \ln(x-3) = \ln(x+15)$$

Your work:

$\ln(x+1) + \ln(x-3) = \ln(x+15)$

$\ln(x+1)(x-3) = \ln(x+15)$

$\therefore (x+1)(x-3) = (x+15)$

$x^2 - 2x - 3 = x + 15$

$x^2 - 3x - 18 = 0$

$(x-6)(x+3) = 0$

$x - 6 = 0$ $\quad \boxed{x = 6}$

$x + 3 = 0$ $\quad \boxed{x = -3}$

Your answer: $x = \cancel{-3}, 6$

not in the domain

1.5

6. (2 points) For which value of the parameter a is the following function continuous at $x = 5$? Justify your answer by explaining what is required for continuity and solving for it.

$$f(x) = \begin{cases} a + \cos(\pi x) & \text{if } x < 5 \\ \frac{x}{2} - 4 & \text{if } x \geq 5. \end{cases}$$

Your work:

1) a function is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and equals $f(a)$

2) $\lim_{x \to 5} f(x)$ exists if

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x)$

$\lim_{x \to 5^-} a + \cos(\pi x) = \lim_{x \to 5^+} \frac{x}{2} - 4$

$a + \cos(5\pi) = \frac{5}{2} - 4$

$a - 1 = \frac{5}{2} - 4$

$a = \frac{5 - 8 + 2}{2}$

$\boxed{a = \frac{-1}{2}}$

3) $\lim_{x \to 5} f(x)$ must equal $f(5)$

Check

Left side $= \lim_{x \to 5} f(x)$

$= \lim_{x \to 5} \frac{x}{2} - 4$

$= \frac{5}{2} - 4$

$= \frac{-3}{2}$

Right side $= \frac{x}{2} - 4$

$= \frac{5}{2} - 4$

$= \frac{5 - 8}{2}$

$= \frac{-3}{2}$

LS = RS

nice.

$a = \boxed{-\frac{1}{2}}$

2

7. (7 points) The population of fish in Fisher's Pond grow at a steady rate annually, but fishing is so popular that, without restocking, the population would die out. Therefore Fisher's Pond is restocked with fish each spring. Using historical data, we declare that DTDS modeling the population of fish, with x_t representing the average number of fish per m^2 of surface area in year t, is given by

$$x_{t+1} = 0.8x_t + 4.8.$$

(a) (1 point) Give the updating function f for this DTDS. $f(x) =$ $\boxed{0.8x + 4.8}$ ✓

(b) (1 point) Find the fixed point x^* of this DTDS.

$x^* = f(x^*)$

$x^* = 0.8x^* + 4.8$

$x^* - 0.8x^* = 4.8$

$0.2x^* = 4.8$

$\boxed{x^* = 24}$

$x^* =$ $\boxed{24}$ ✓

(c) (1 point) Suppose that in year zero there were 15 fish/m^2. Give the general solution formula to this DTDS.

x_0

$x_t = 0.8^t(x_0 - x^*) + x^*$

$= 0.8^t(-9) + 24$

General solution formula: $\boxed{x_t = 0.8^t(-9) + 24}$ ✓

4

(d) (1 point) Find the number of fish per m^2 after two years.

$x_2 = 0.8^2(-9) + 24$

$= 18.24$

average fraction correct

Your answer: $\boxed{x_2 = 18 \text{ fish}}$

(e) (3 points) Determine the (whole) number of years necessary until the number of fish per m^2 is within 0.9 of the fixed point, that is, until $|x_t - x^*| < 0.9$. **Show your work. Your answer must be clear and well-justified to earn full marks.**

$x^* = 24$

$23.1 < x_t < 24.9$ ✓

~~$x_t = 0.8^t(-9) + 24$~~

~~$23.1 = 0.8^t(-9) + 24$~~

~~$-0.9 = 0.8^t(-9)$~~

~~$0.1 = 0.8^t$~~

~~$\log 0.1 = t \log 0.8$~~

~~$t = \frac{\log 0.8}{\log 0.1}$~~ ~~$t = 0.097$~~

$|x_t - x^*| < 0.9$

$|0.8^t(-9) + \cancel{24} - \cancel{24}| < 0.9$

$|0.8^t(0.9)| < 0.9$

X ?

Case 1

$+(0.8^t(0.9)) < 0.9$

$0.8^t < 1$

$t\log(0.8) < \log 1$

? $\boxed{t > 0}$ since $\log(0.8) < 0$.

Case 2

$-(0.8^t(0.9)) < 0.9$

$0.8^t < -1$ never true

Can't take the log of a negative number

the closest whole number of years is 0 ??

2

8. (5 points) The DTDS $x_{t+1} = 0.3x_t(9.1 - x_t)$ models a certain population.

(a) (2 points) Solve for all fixed points of the DTDS.

Your work:

$X_{t+1} = 0.3X_t(9.1 - X_t)$

$X^* = 0.3X^*(9.1 - X^*)$

$X^* = 2.73X^* - 0.3X^{*2}$

$0 = 2.73X^* - X^* - 0.3X^{*2}$

$0 = 1.73X^* - 0.3X^{*2}$

$0 = X^*(1.73 - 0.3X^*)$

$\boxed{X^* = 0}$

$0 = 1.73 - 0.3X^*$

$-1.73 = -0.3X^*$

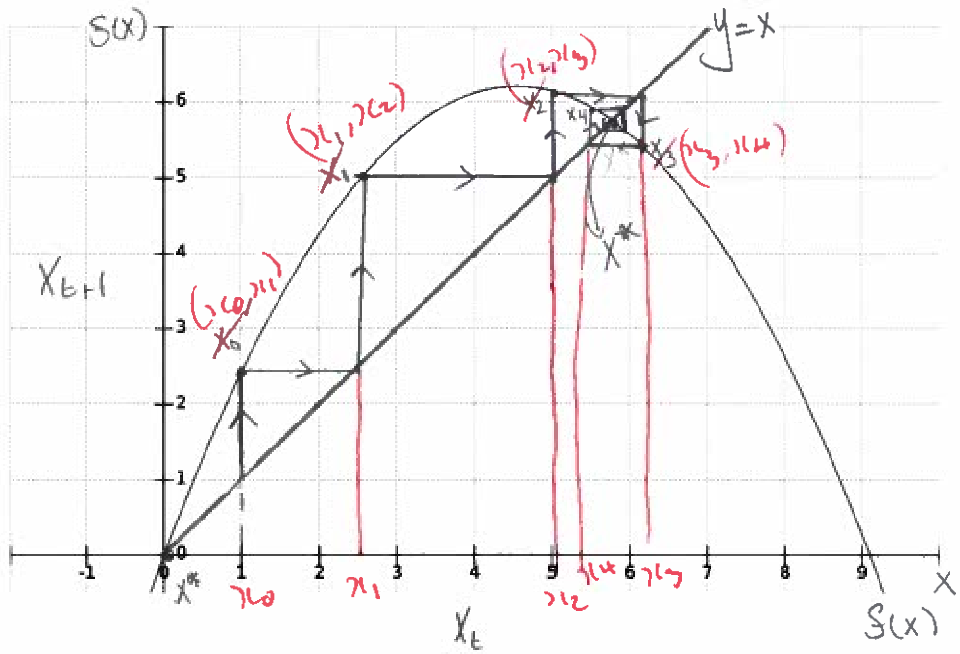
$X^* = \frac{-1.73}{-0.3}$

$\boxed{X^* = 5.8}$

2

Your answer: $X^* = 0, 5.8$

(b) (2 points) The graph of the updating function of this DTDS is given below. Suppose the initial value is $x_0 = 1$. Draw a cobweb diagram on the graph below for this DTDS with at least 4 steps. Label the axes, the functions, the fixed points and the points $x_0 \cdots x_4$.



✓ cobweb

✗ labels

1.5

(c) (1 point) Write a sentence to explain what happens in the long term if $x_0 = 1$. Your sentence should include the word "stable" or "unstable", as well as the exact fixed point from (a) which is relevant.

if $X_0 = 1$, the fixed point $X^* = 5.8$ is stable since the nearby X_0 fall towards it.

1

9. (4 points) Decide if the following limits exist. For each one, if it exists, evaluate the limit exactly using algebraic methods, showing all your steps. If it does not exist, justify your answer clearly using mathematical reasoning.

(a) (2 points) $\lim_{x \to 4} \frac{x-4}{3-\sqrt{25-x^2}}$

$$\lim_{x\to 4} \frac{x-4}{3-\sqrt{25-x^2}}$$

$$= \lim_{x\to 4} \frac{x-4}{3-\sqrt{25-x^2}} \times \frac{3+\sqrt{25-x^2}}{3+\sqrt{25-x^2}}$$

$$= \lim_{x\to 4} \frac{3x-12+x\sqrt{25-x^2}-4\sqrt{25-x^2}}{9-(25-x^2)}$$

$$= \frac{\cancel{3(4)-12}+4\sqrt{25-(4)^2}-4\sqrt{25-(4)^2}}{9-(25-(4)^2)}$$

$$= \frac{4\sqrt{25-16}-4\sqrt{25-16}}{-32 \; \times \; 0}$$

$$= \frac{0}{-32} = 0$$

OR

$$\lim_{x\to 4} \frac{x-4}{3-\sqrt{25-x^2}}$$

$$= \frac{0}{3-\sqrt{25-16}} = \cancel{x} \; \frac{0}{0}$$

$\to 0$

Why did i do all that?

Your answer: $\boxed{0}$

(b) (2 points) $\lim_{x \to \infty} \frac{3x^2}{\sqrt{4x^4+3}}$

$$\lim_{x\to\infty} \frac{3x^2}{\sqrt{4x^4+3}} \times \frac{\sqrt{4x^4+3}}{\sqrt{4x^4+3}}$$

$$= \lim_{x\to\infty} \frac{3x^2\sqrt{4x^2+3}}{4x^4+3} \times \frac{\frac{1}{x^4}}{\frac{1}{x^4}}$$

$$= \lim_{x\to\infty} \frac{\cancel{\frac{3}{x^2}} \cancel{\frac{\sqrt{4x^2+3}}{x^4}}}{4+\cancel{\frac{3}{x^4}}}$$

$\to 0$ $\to 0$

$$= \lim_{x\to\infty} \frac{0}{4}$$

Your answer: $\boxed{0}$ X

simplify: $\frac{1}{x^4}\sqrt{4x^4+3} = \sqrt{\frac{4x^4+3}{x^8}}$

$$\frac{ab}{x^4} \neq \frac{a}{x^4}\,\frac{b}{x^4}$$

$$\frac{a+b}{x^4} = \frac{a}{x^4} + \frac{b}{x^4}$$